**ITEMS Digital Module 11:**

**Bayesian Psychometric Modeling**

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**This document contains only the core content slides from the module. In the digital module all slides can be accessed individually.**

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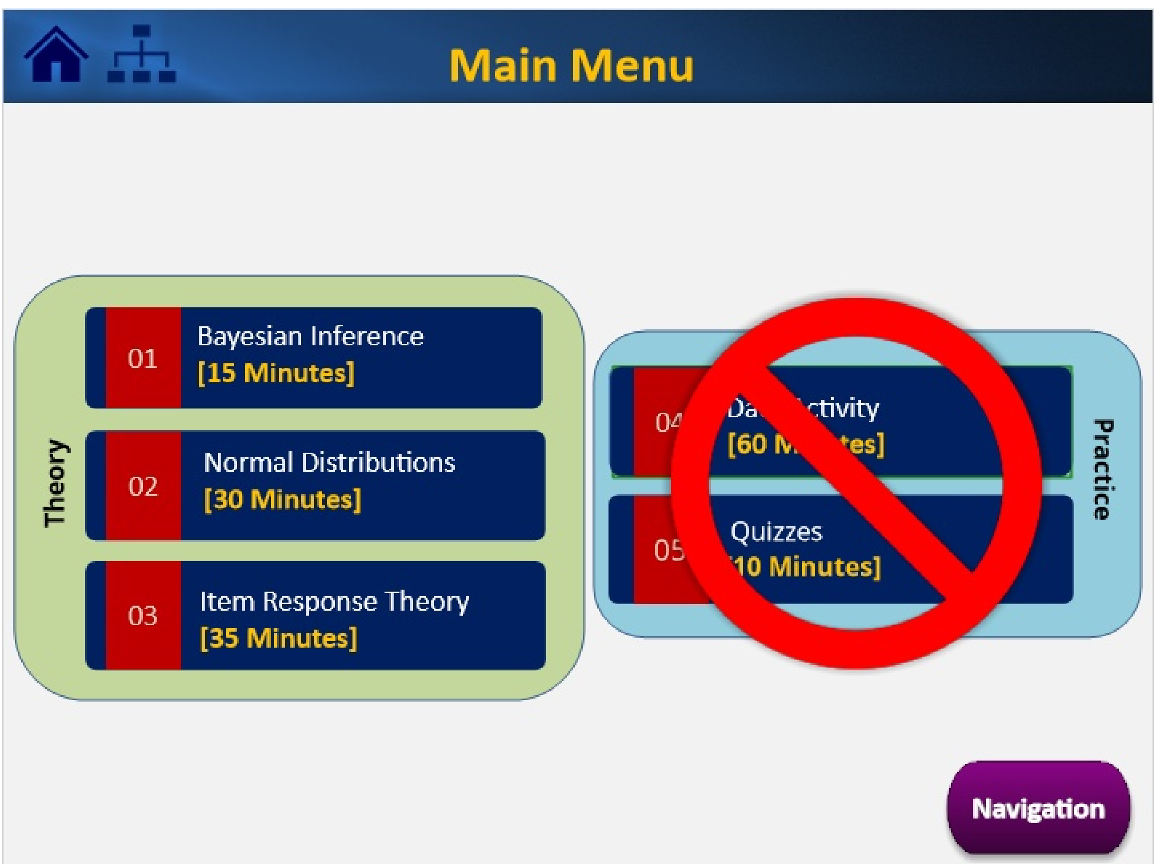
## Module Organization

The module starts with an introductory section that leads to the main menu from which learners can select individual theory and application sections; this slide deck contains the slides for the three content sections only:

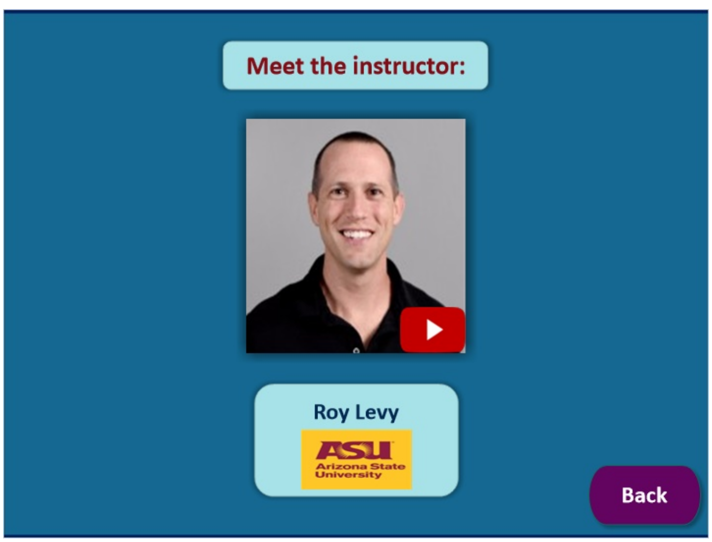


# DM11 SLIDES (Bayesian Psychometrics, Version 1.0)

## 1. Module Overview

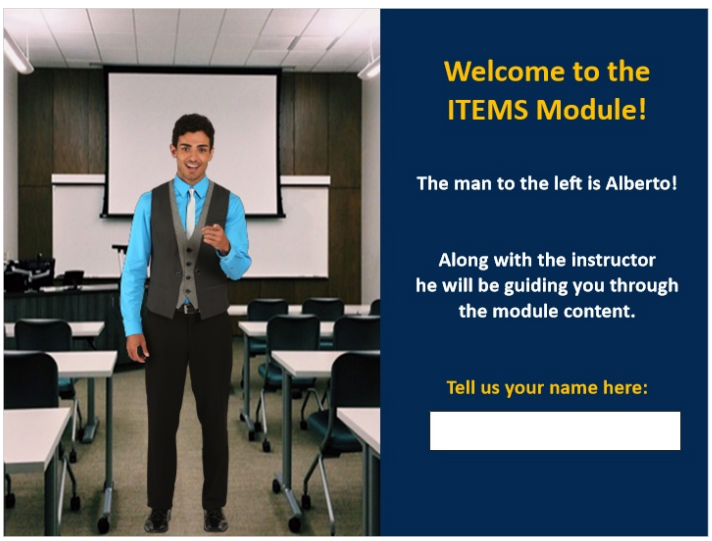
### *1.1 Module Cover*


### *1.2 Instructor*

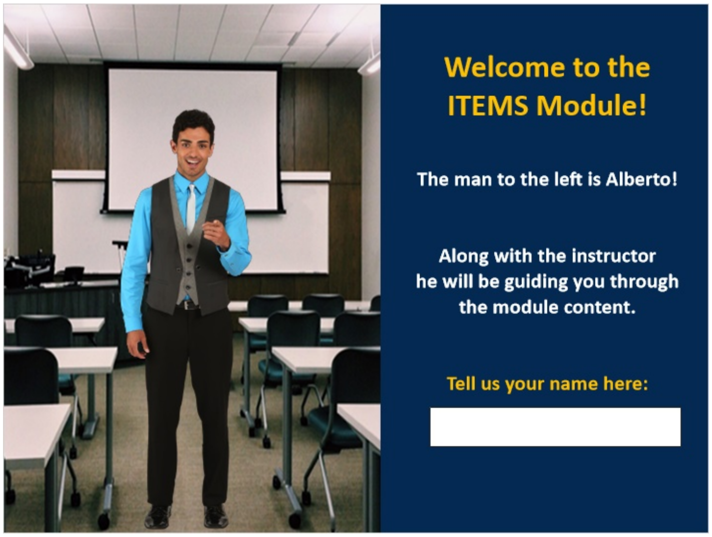


## 1.3 Designers

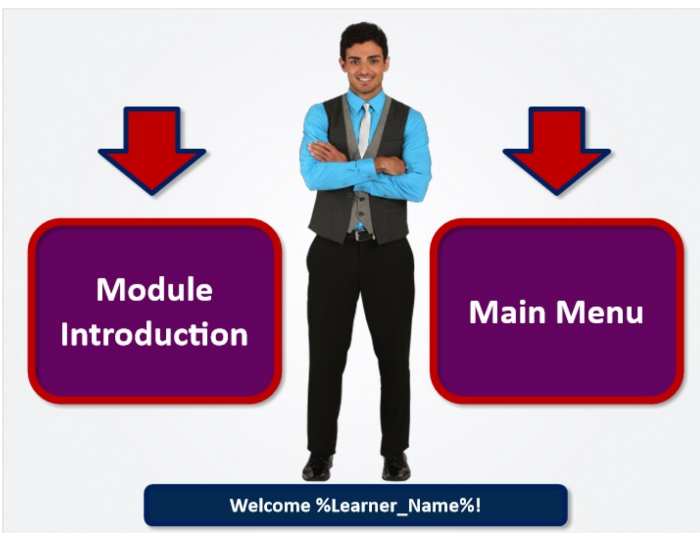

## 1.4 Welcome



## Untitled Layer 1 (Slide Layer)




## *1.5 Path Choice*

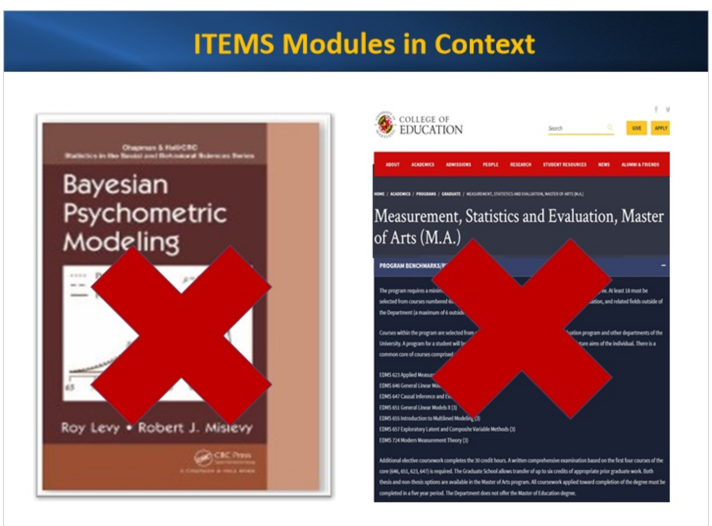


## 1.6 Overview

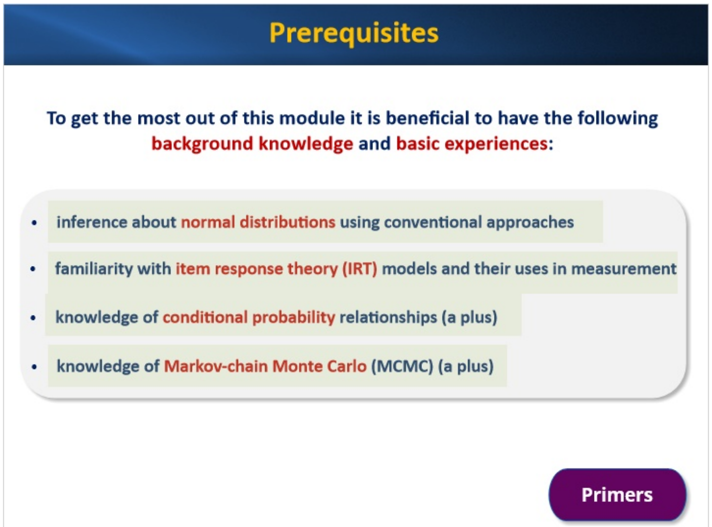

## 1.7 Target Audience

## 1.8 Expecations (I)

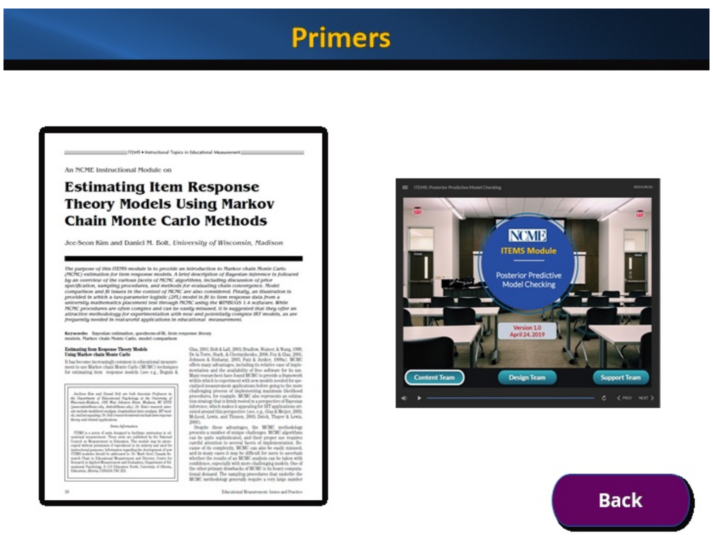

## 1.9 Expectations (II)



## 1.10 Prerequisites

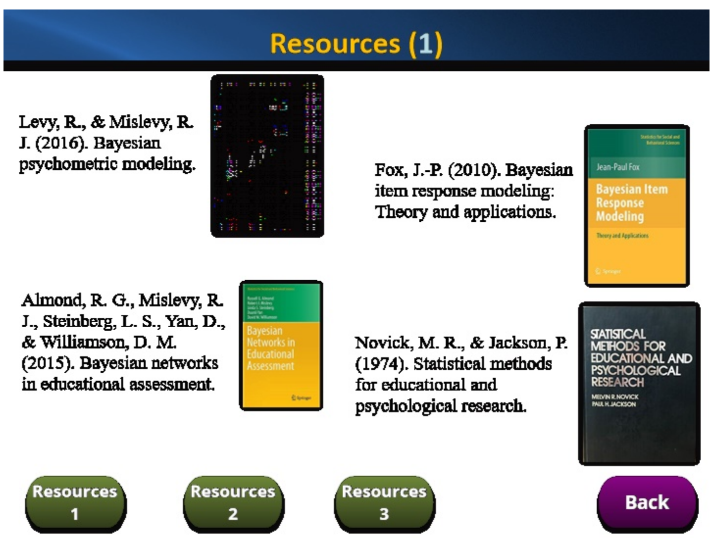



### Primers (Slide Layer)

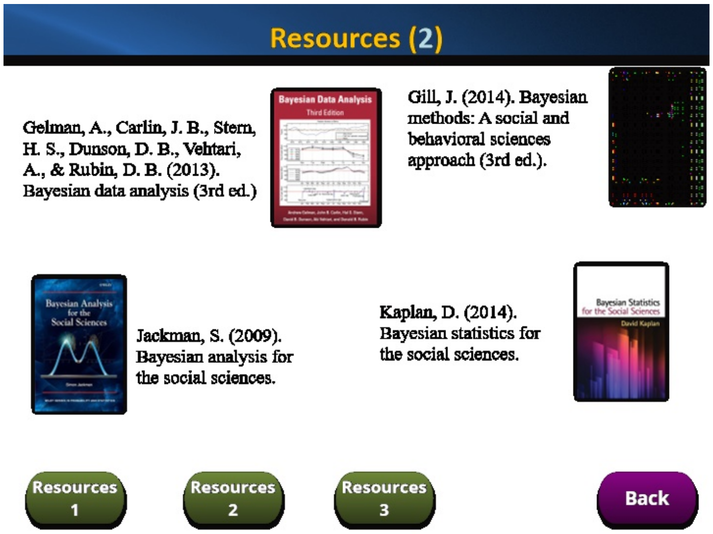


## *1.11 Resources*

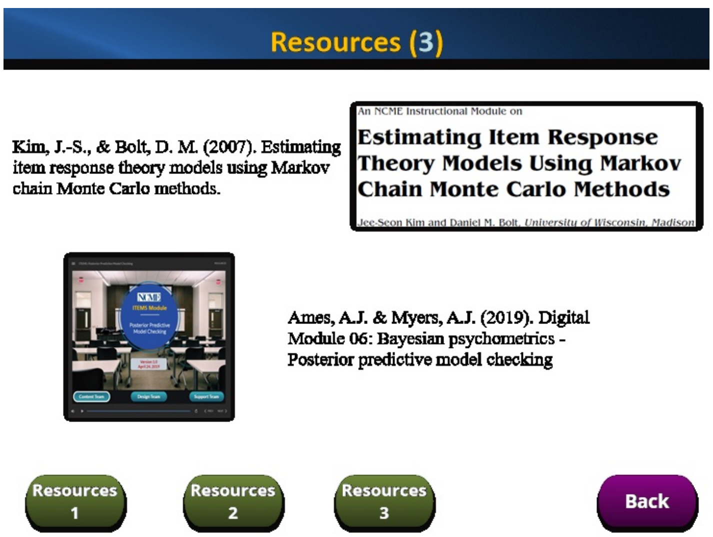

## Resources 1 (Slide Layer)



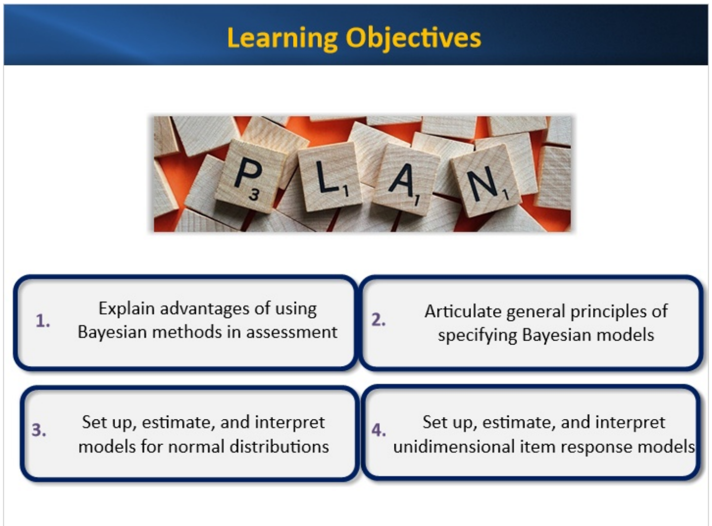
## Resources 2 (Slide Layer)



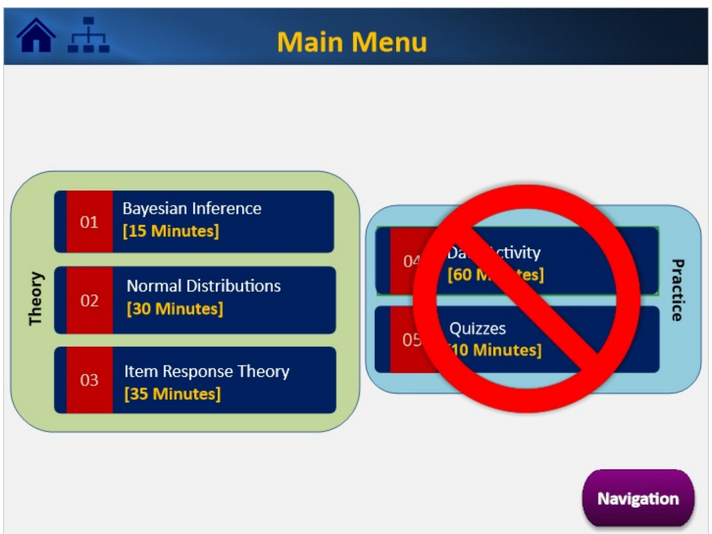

## Resources 3 (Slide Layer)



## 1.12 Learning Objectives



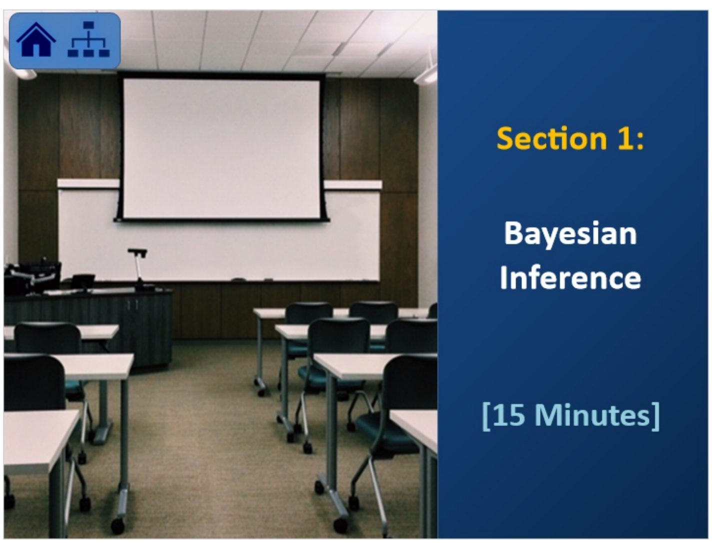

## 1.13 Main Menu



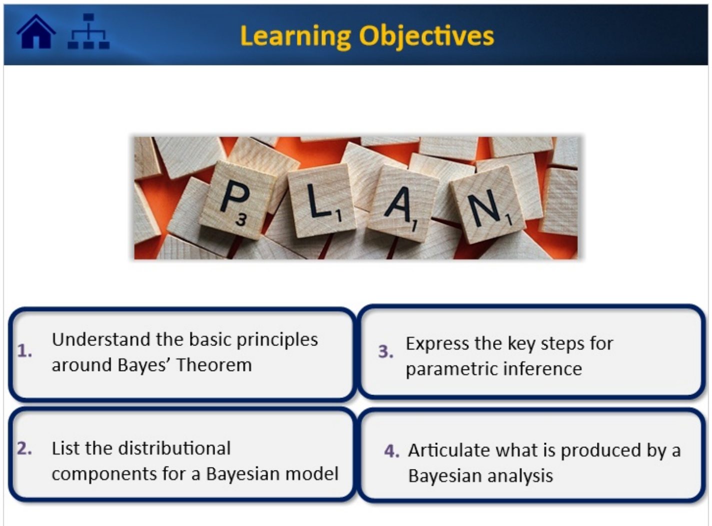
**Navigation Help (Slide Layer)**


## 2. Section 1: Bayesian Inference

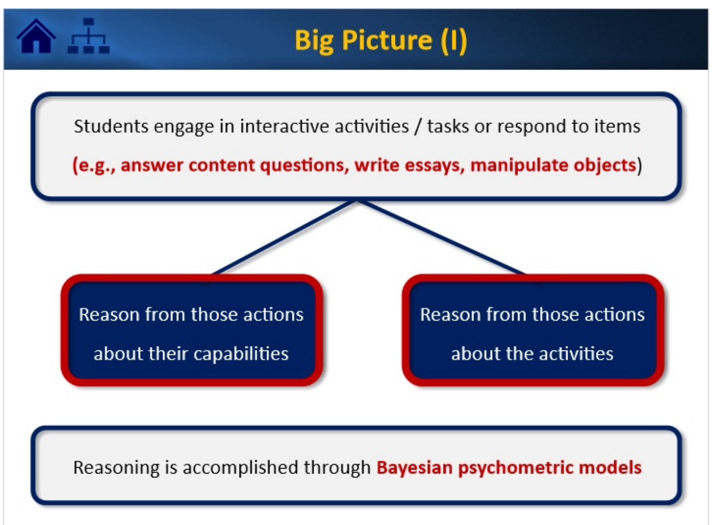
### *2.1 Cover: Section 1 Why Bayes*



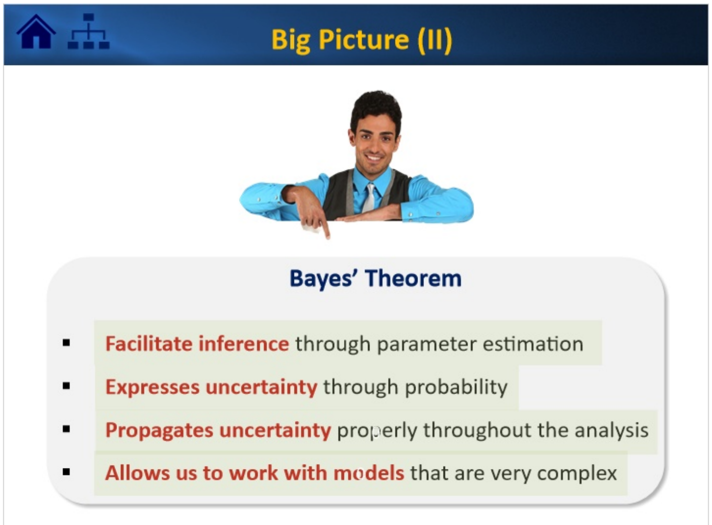
## 2.2 Learning Objectives



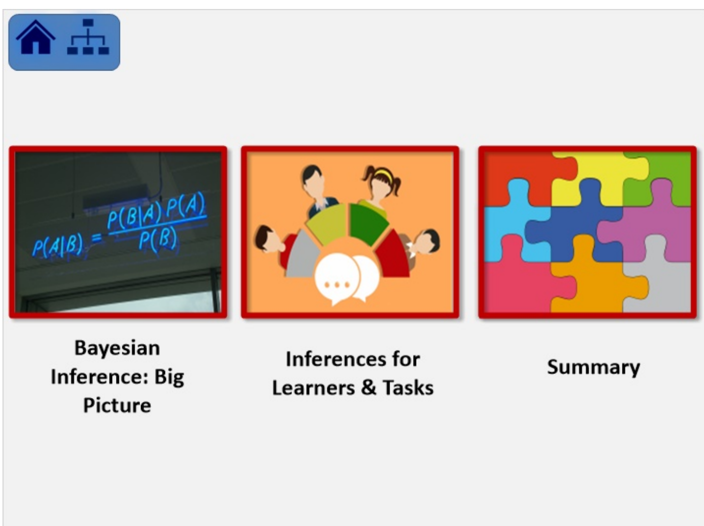

## 2.3 Big Picture (I)



### 2.4 Big Picture (II)

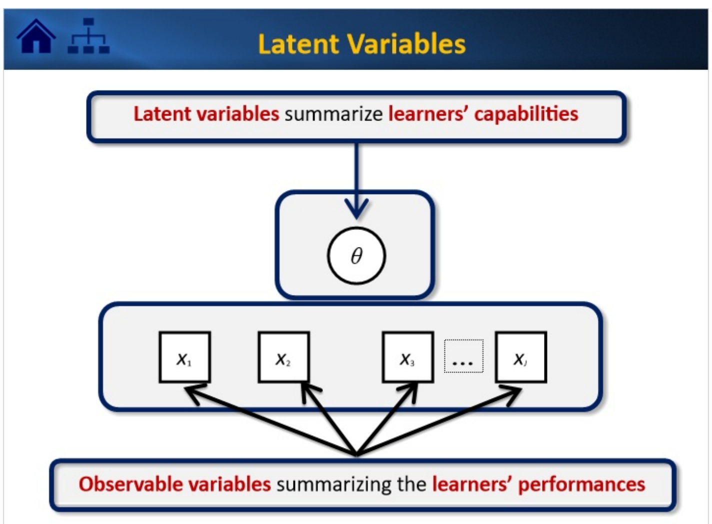



### 2.5 Topic Selection

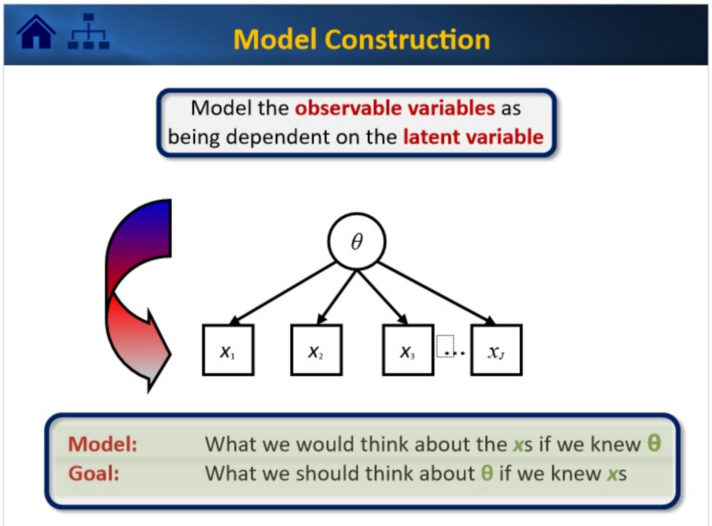


## *2.6 Bookmark: Bayes' Theorem*

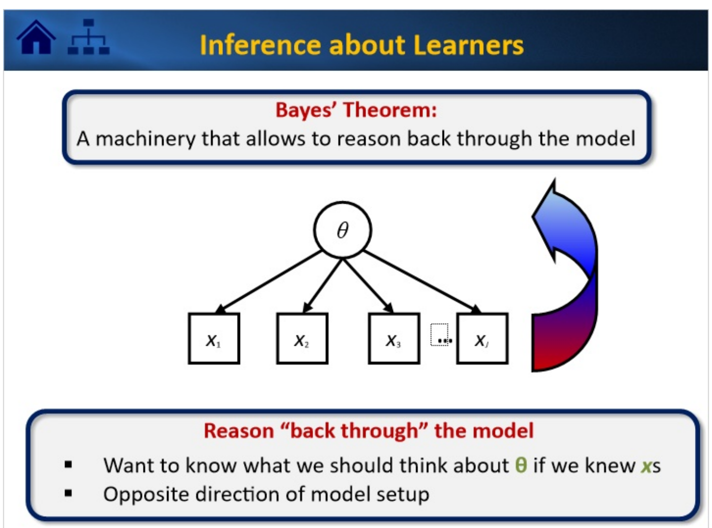

## *2.7 Latent Variables*



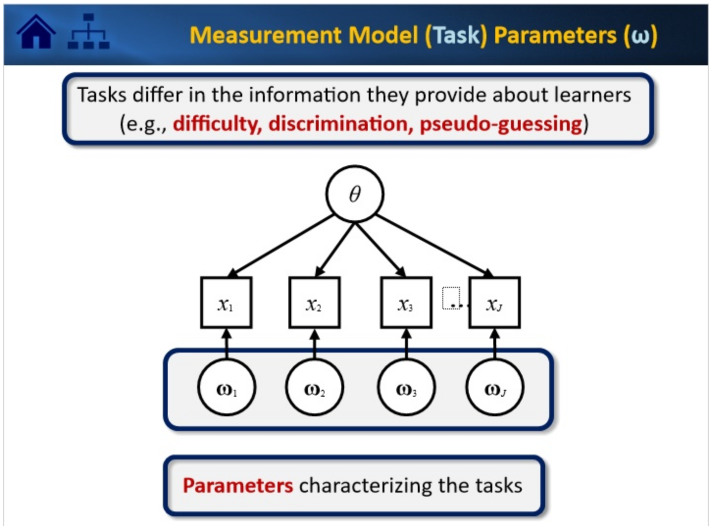
## 2.8 Model Construction



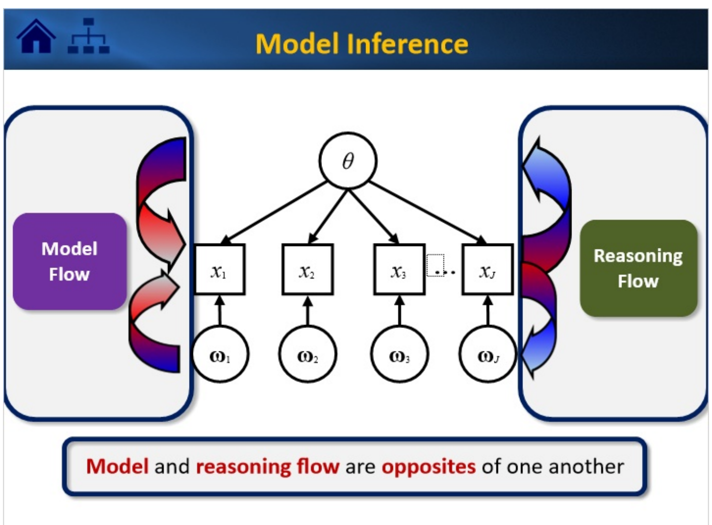

## 2.9 Inference about learners



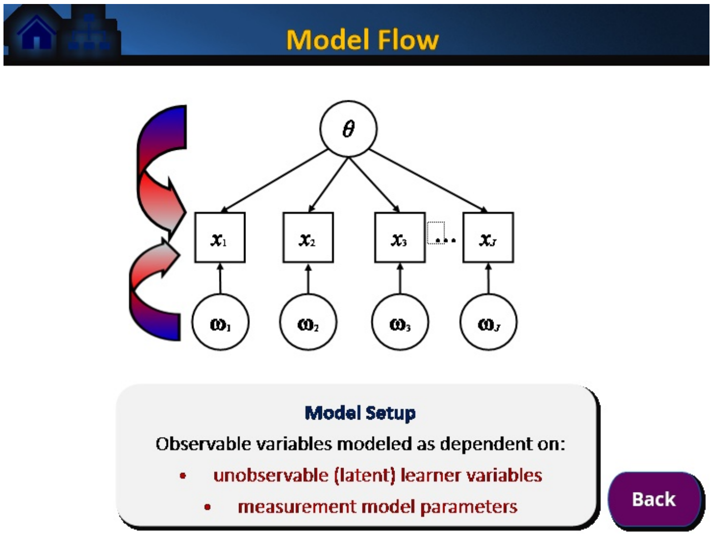
## *2.10 Task Parameters*



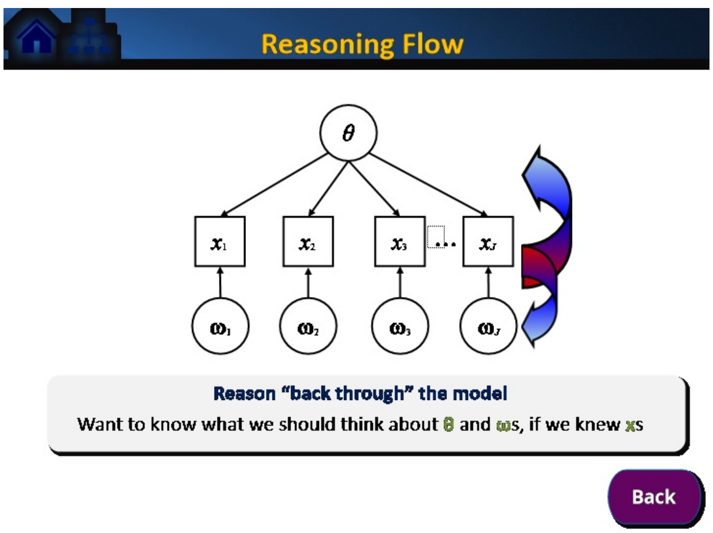

## *2.11 Model Inference*



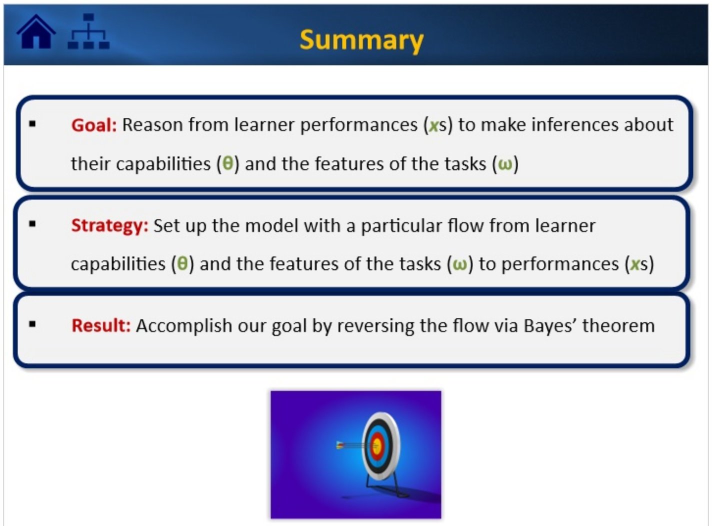
## Model flow (Slide Layer)



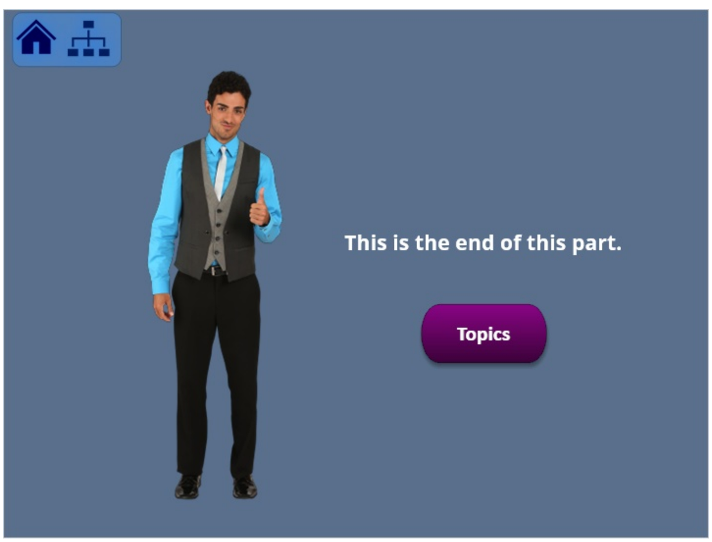

## Reasoning flow (Slide Layer)



## *2.12 Summary*




## *2.13 Bookend: Bayes' Theorem*



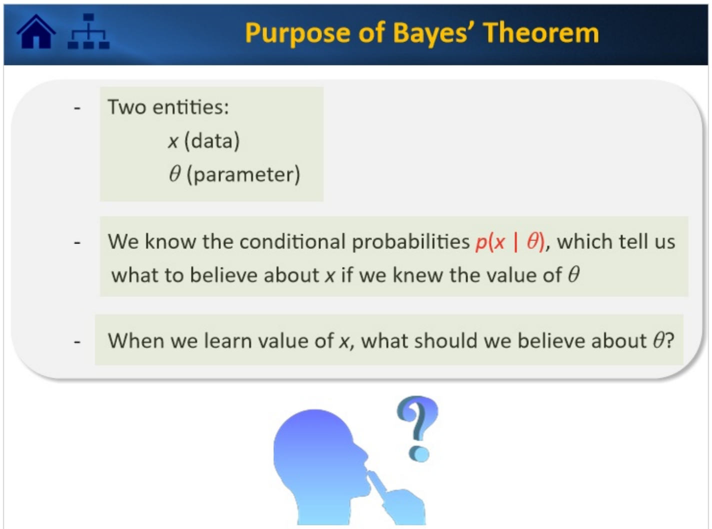
## 2.14 Bookmark: Learners & Tasks


## 2.15 Bayes' Theorem (I)

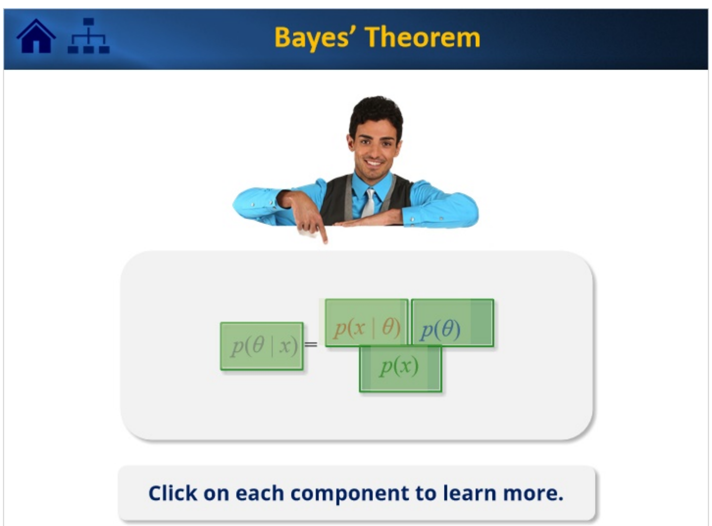
**Notation**

| | |
|---|---|
| $\theta$ | A parameter (e.g., a population mean) |
| $p(x \mid \theta)$ | Conditional probability of data ($x$) given the parameter ($\theta$)<br>Likelihood of the parameter ($\theta$) given data ($x$) |
| $p(\theta)$ | The ***prior*** distribution for $\theta$ |
| $p(\theta \mid x)$ | The ***posterior*** distribution for $\theta$ given $x$ |

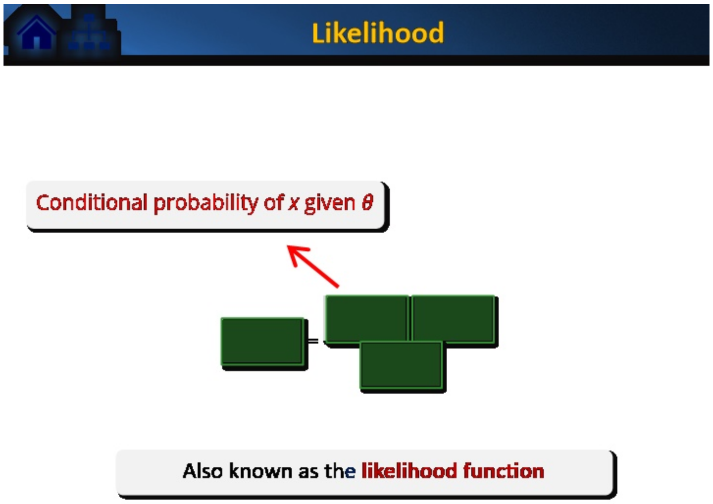
## 2.16 Bayes' Theorem (I)



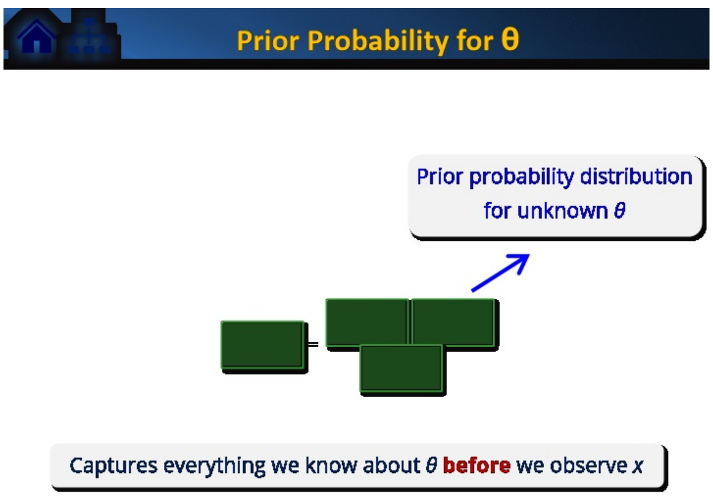

## 2.17 Bayes Theorem



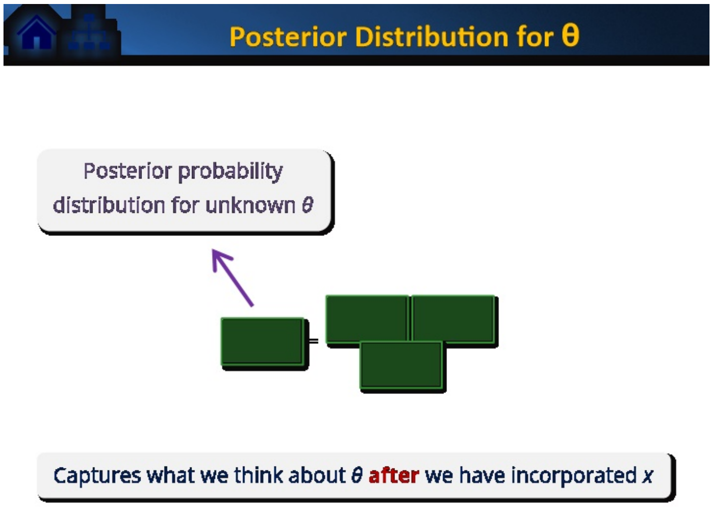
## Likelihood (Slide Layer)



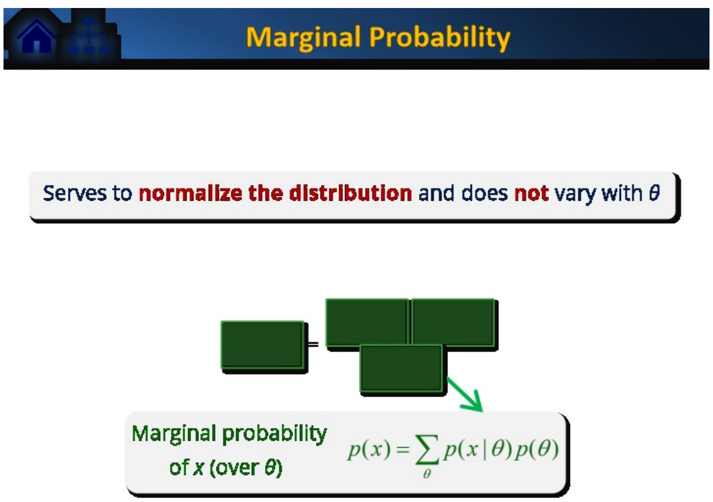

## Prior (Slide Layer)



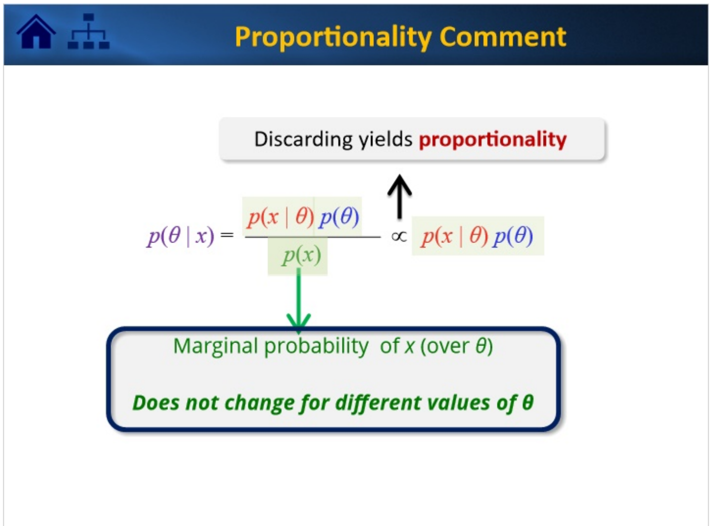
## Posterior (Slide Layer)




## Constant (Slide Layer)

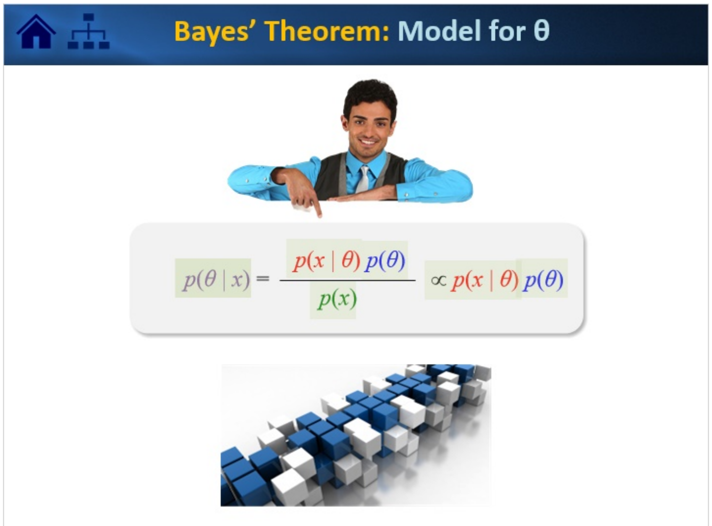


## *2.18 Posterior (II)*

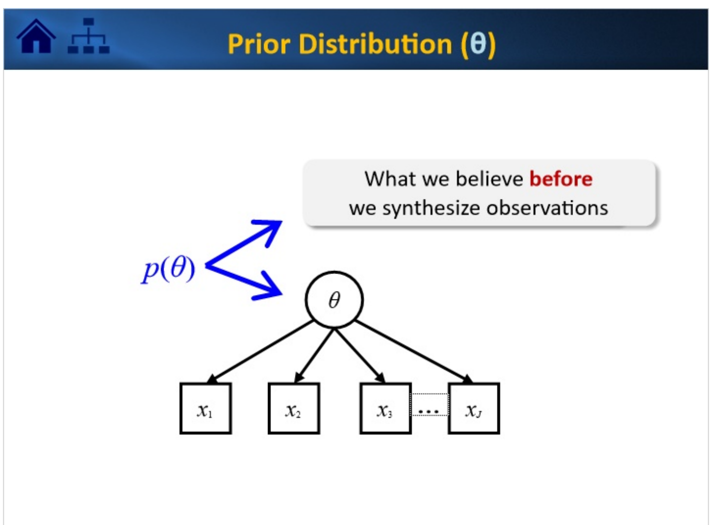



## *2.19 Topic Selection*

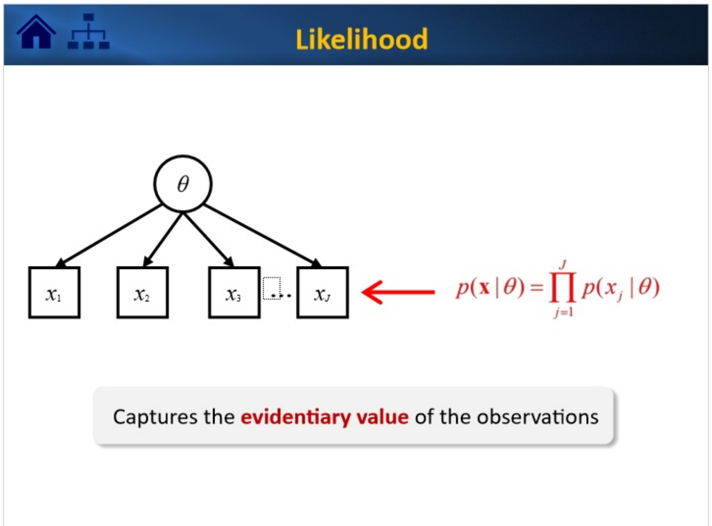
## 2.20 Bayes' Theorem: Model for ϑ



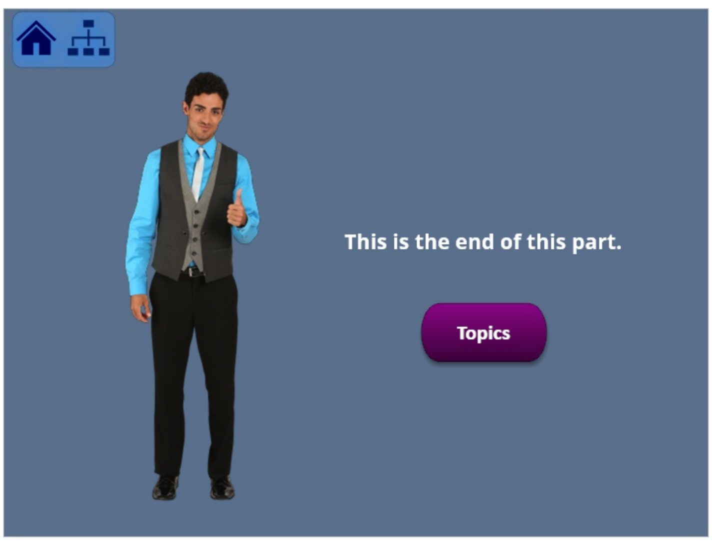

## 2.21 Prior Distribution (ϑ)



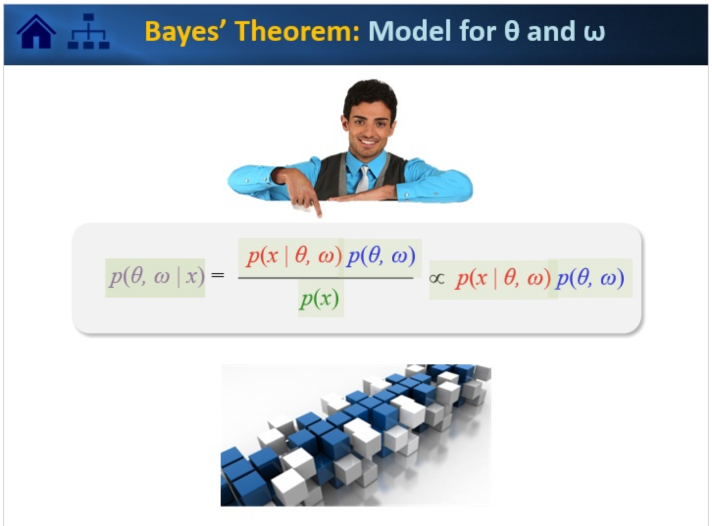
## 2.22 Conditional distribution



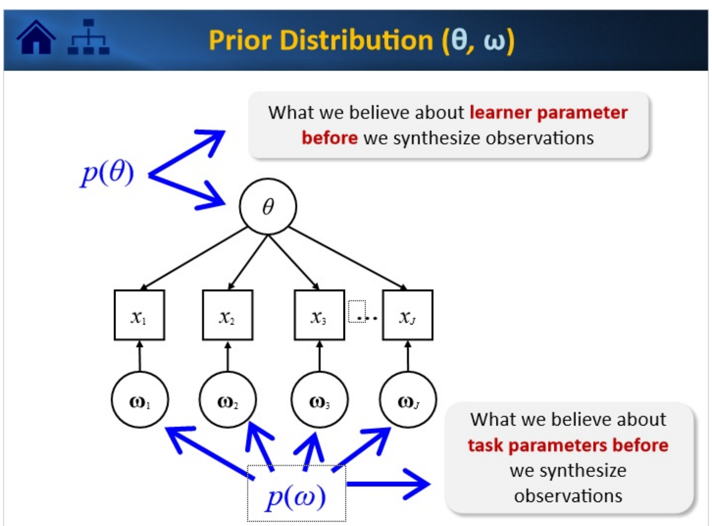

## 2.23 Bookend: Learner Parameters



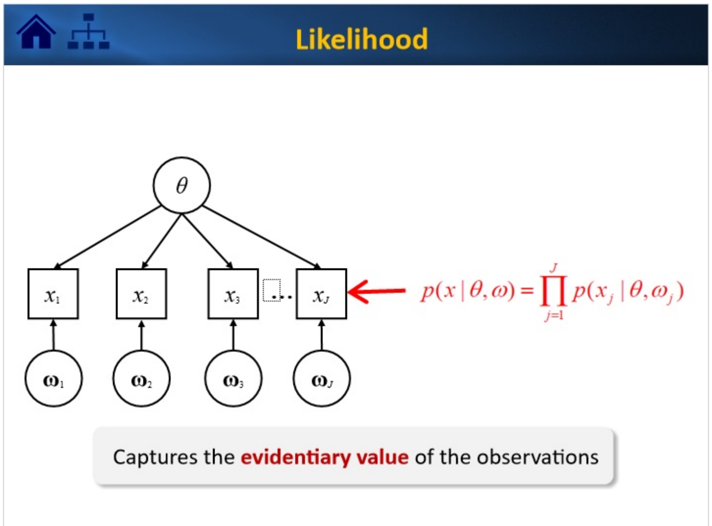
### *2.24 Bayes' Theorem: Model for ϑ and w*



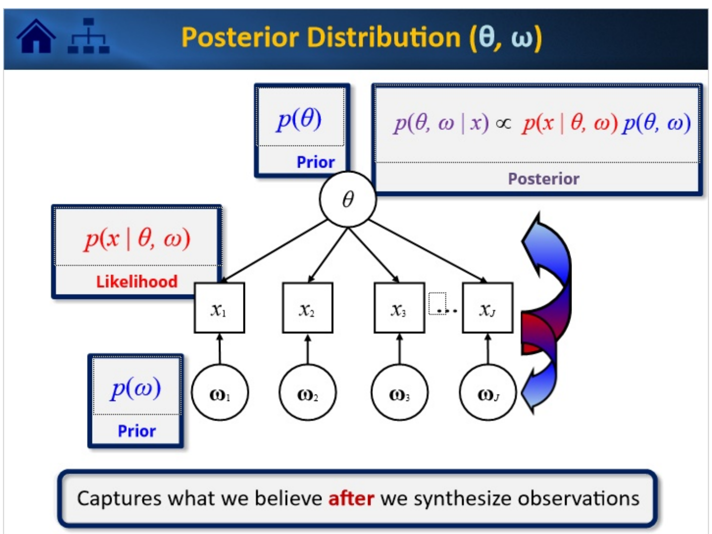

### *2.25 Prior Distribution (ϑ)*



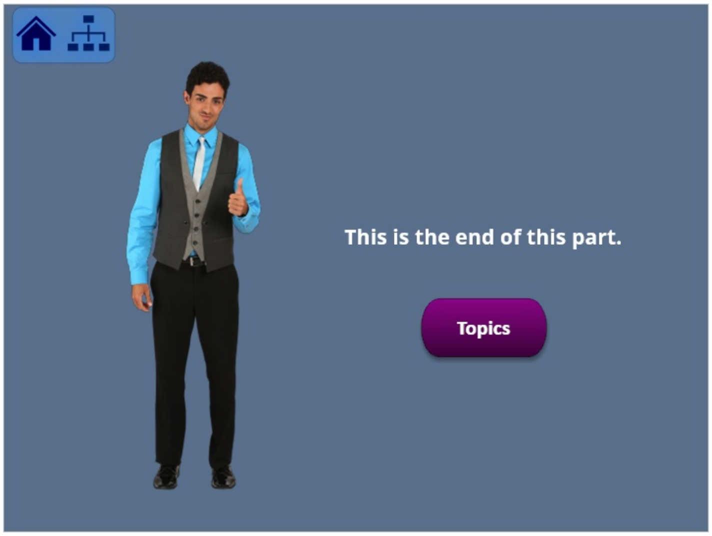
## 2.26 Conditional distribution




## 2.27 Posterior

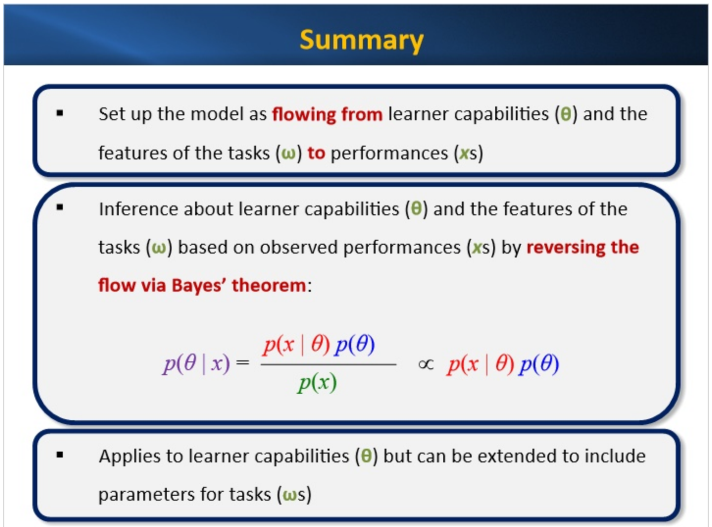


## *2.28 Bookend: Learners & Items*

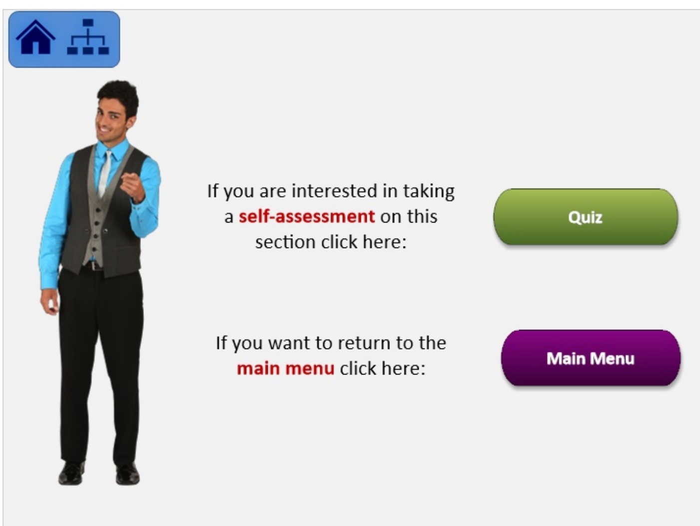



## *2.29 Bookmark: Summary*

## *2.30 Summary*

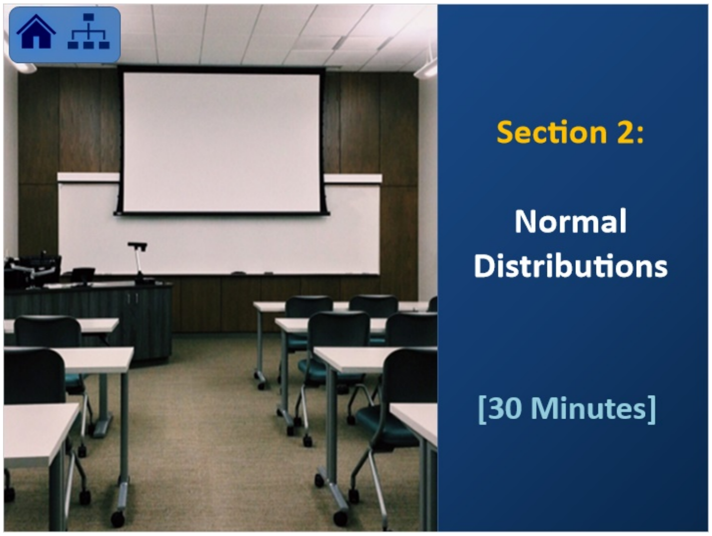



## *2.31 Bookend: Section 1*

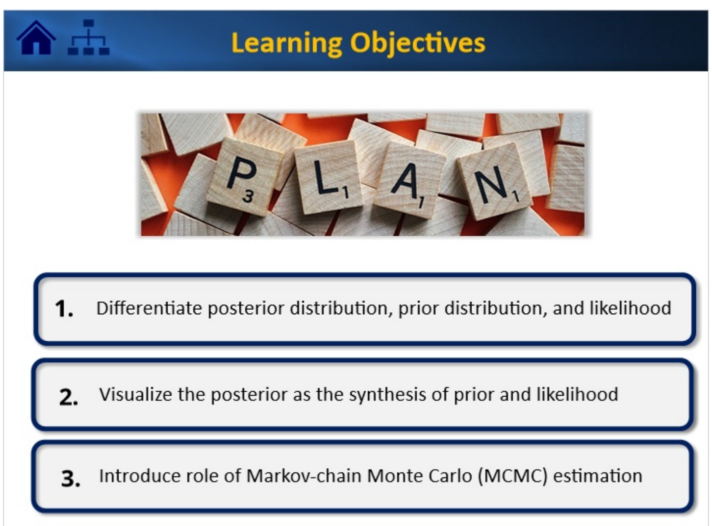


# 3. Section 2: Normal Distributions

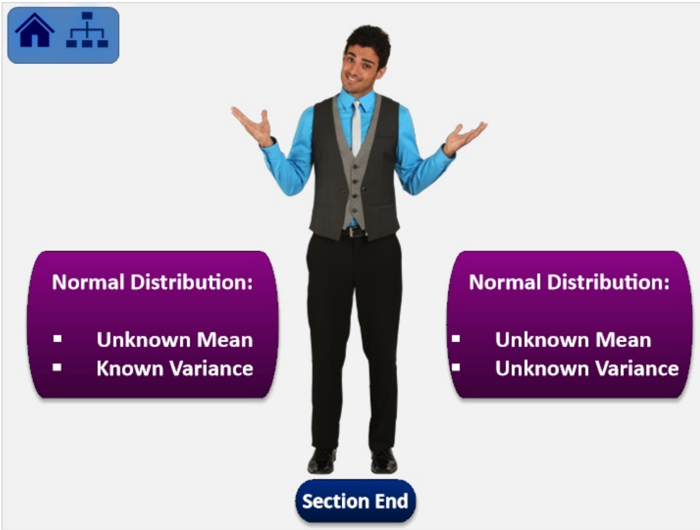
## *3.1 Cover: Section 2 Bayesian Inference*




## *3.2 Learning Objectives*

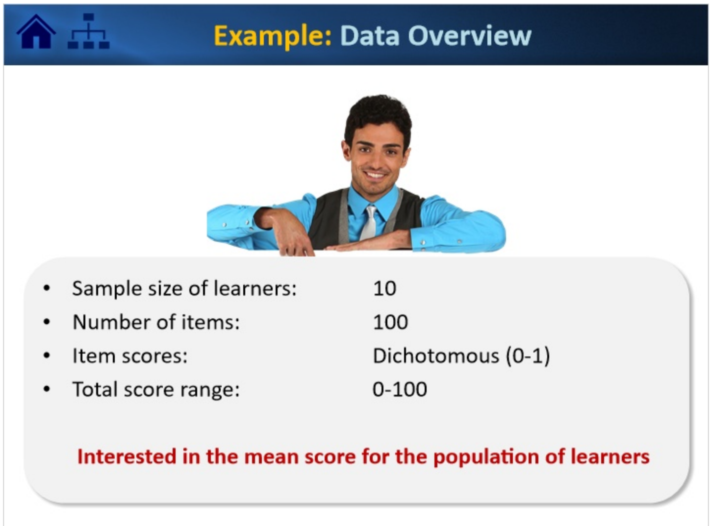


### *3.3 Topic Selection*

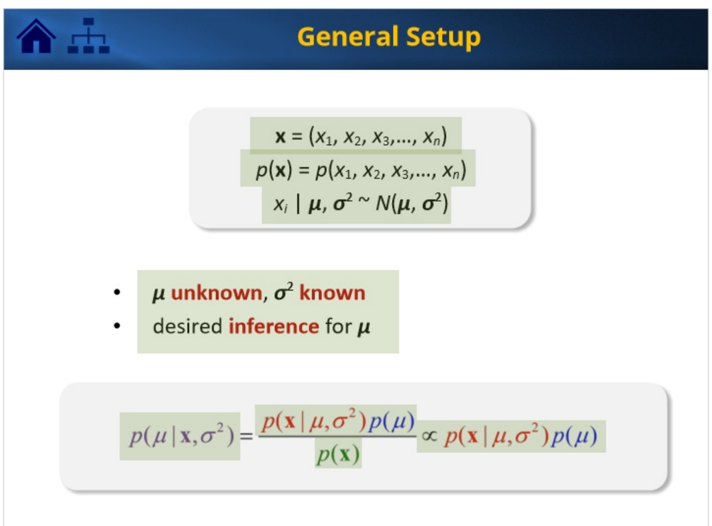



### *3.4 Bookmark: Setup 1*

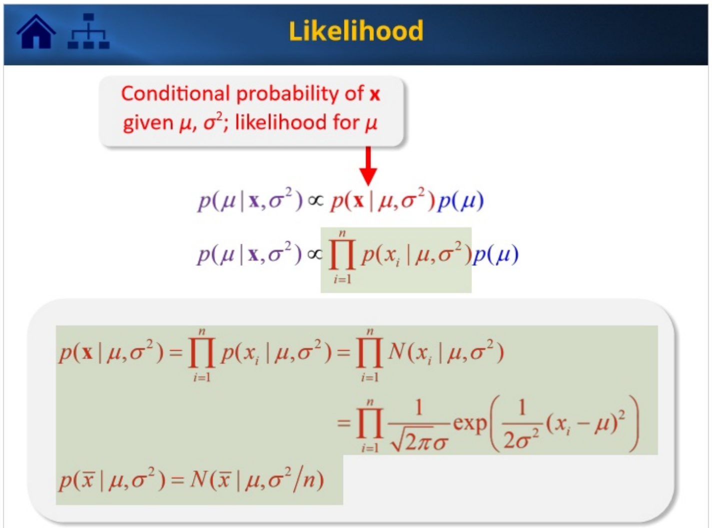
### 3.5 Data Overview



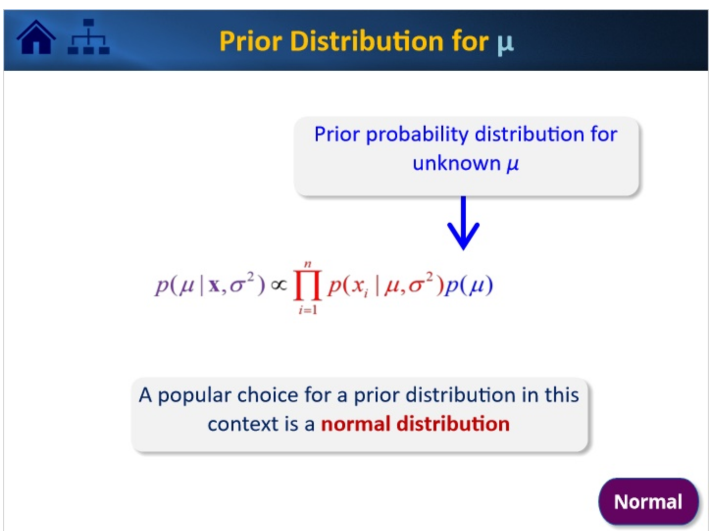

### 3.6 Modeling Setup



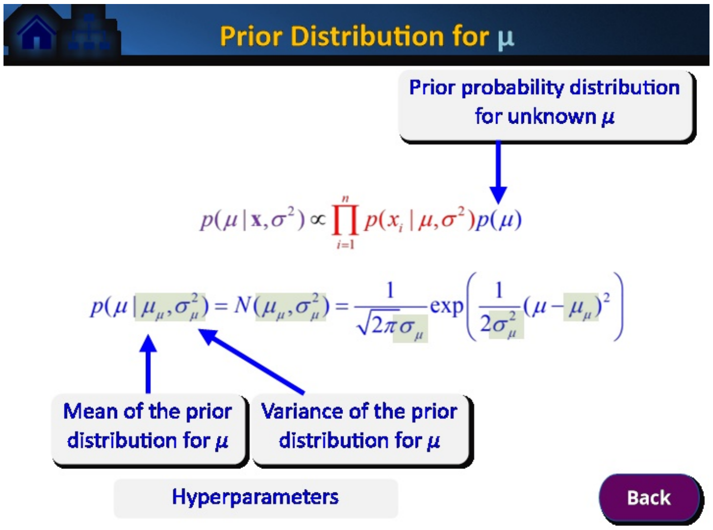
## *3.7 Likelihood*



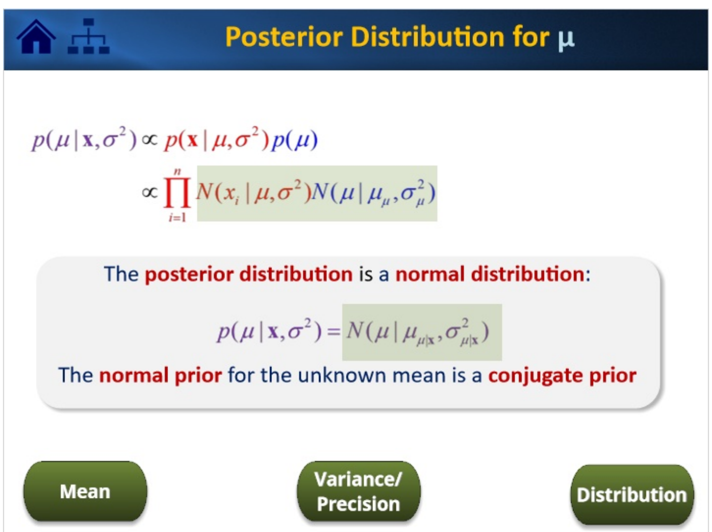

## *3.8 Prior*



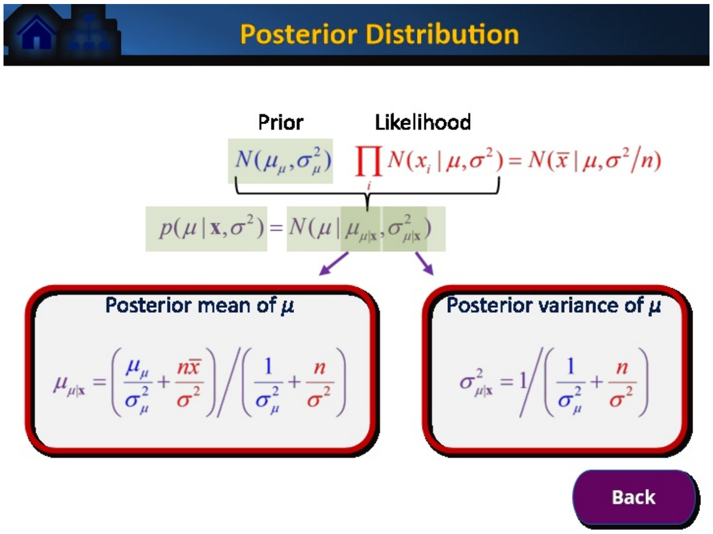
## Normal (Slide Layer)



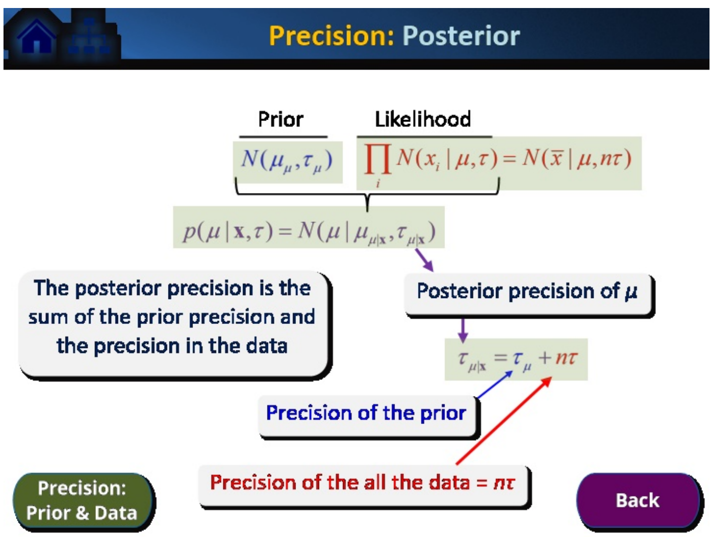

## *3.9 Posterior (Mean)*



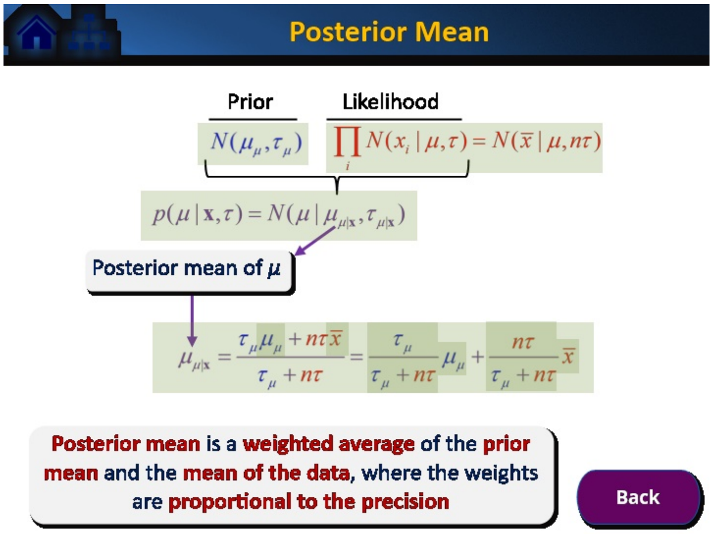
## Distribution (Slide Layer)



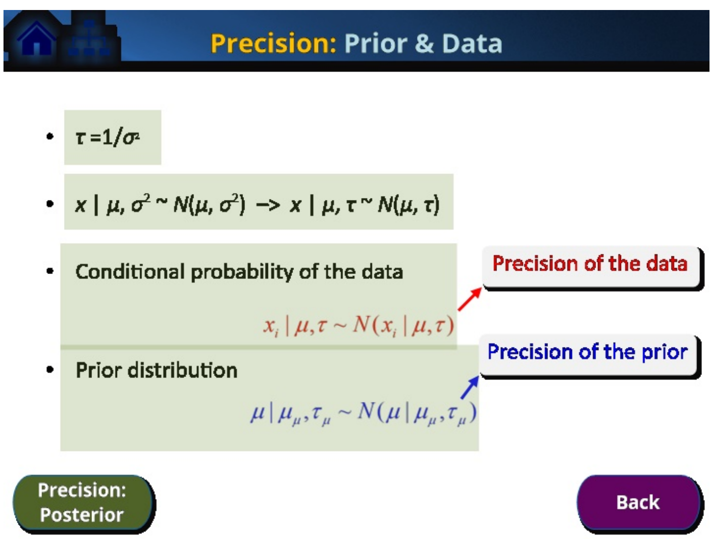

## Precision Posterior (Slide Layer)



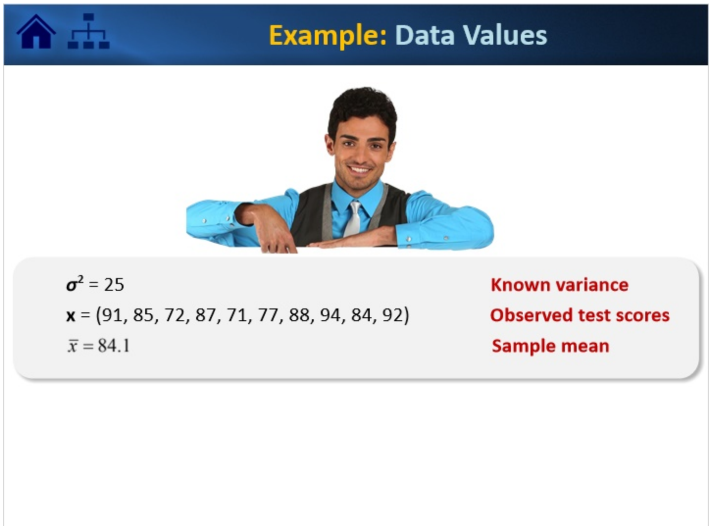
## Mean (Slide Layer)



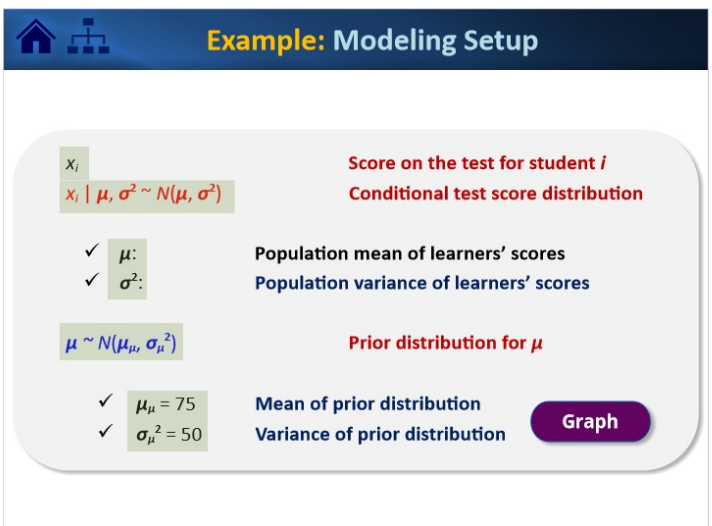

## Precision Prior Data (Slide Layer)



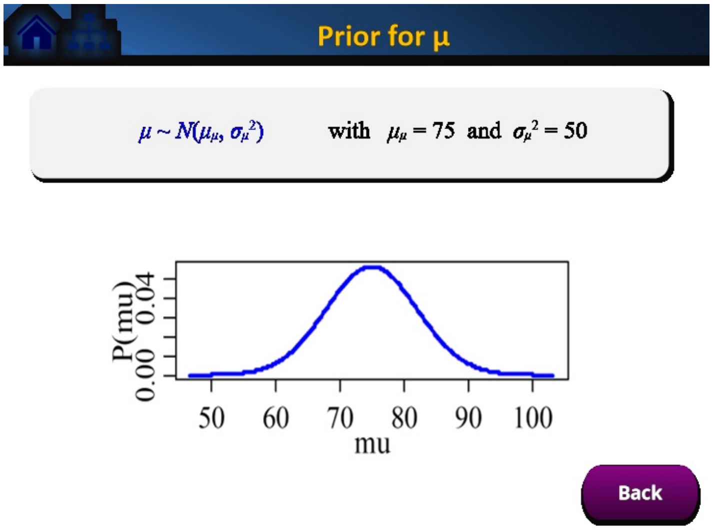
## *3.10 Data Values*



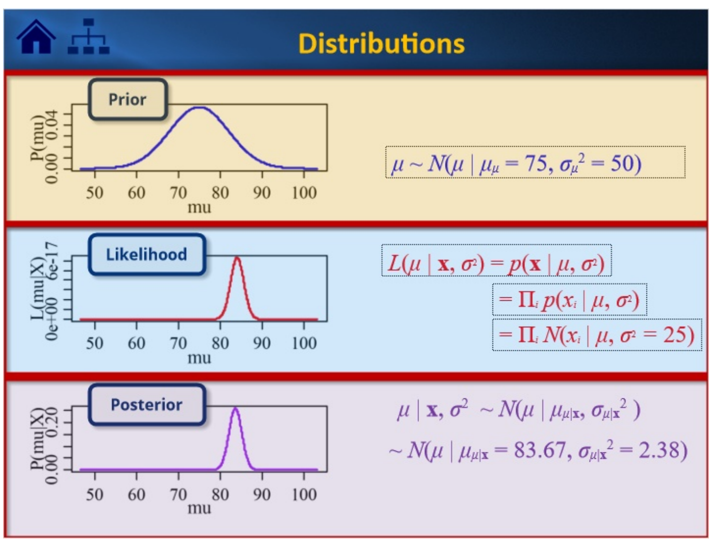

## *3.11 Modeling Setup Revisited*



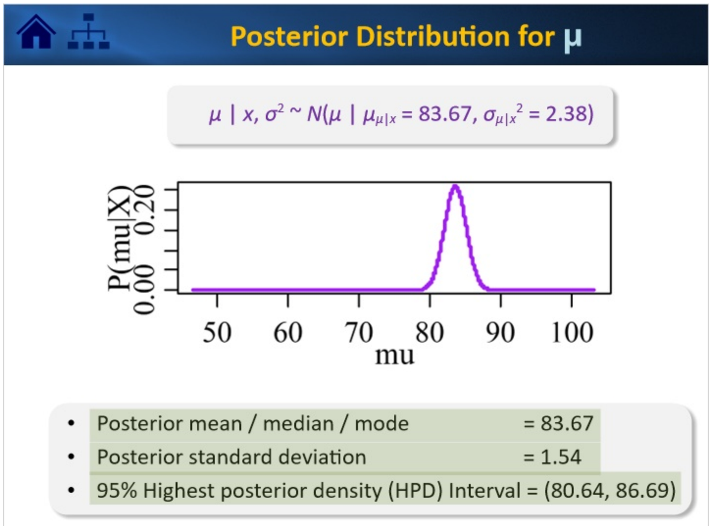
## Graph of Prior Mean (Slide Layer)



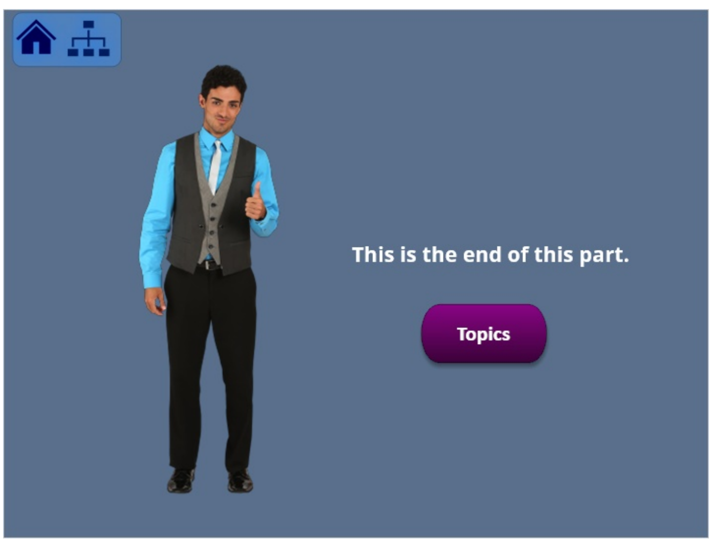

### *3.12 Distributions*



## 3.13 Posterior

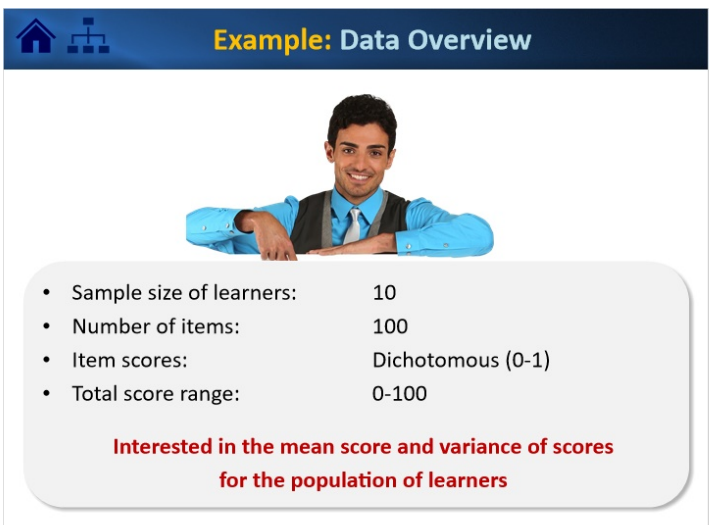



## 3.14 Bookend: Setup 1



## *3.15 Bookmark: Setup 2*


## *3.16 Data Overview*



## *3.17 Modeling Setup*

**General Setup**

$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)$

$p(\mathbf{x}) = p(x_1, x_2, x_3, \ldots, x_n)$

$x_i \mid \boldsymbol{\mu}, \boldsymbol{\sigma}^2 \sim N(\boldsymbol{\mu}, \boldsymbol{\sigma}^2)$

- $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}^2$ **unknown**
- desired **inference** for $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}^2$

$$p(\mu,\sigma^2 \mid \mathbf{x}) = \frac{p(\mathbf{x} \mid \mu,\sigma^2)\, p(\mu,\sigma^2)}{p(\mathbf{x})} \propto p(\mathbf{x} \mid \mu,\sigma^2)\, p(\mu,\sigma^2)$$

## *3.18 Likelihood*

**Likelihood**

Conditional probability of **x** given $\mu$, $\sigma^2$; likelihood for $\mu$

$$p(\mu,\sigma^2 \mid \mathbf{x}) \propto p(\mathbf{x} \mid \mu,\sigma^2)\, p(\mu,\sigma^2)$$

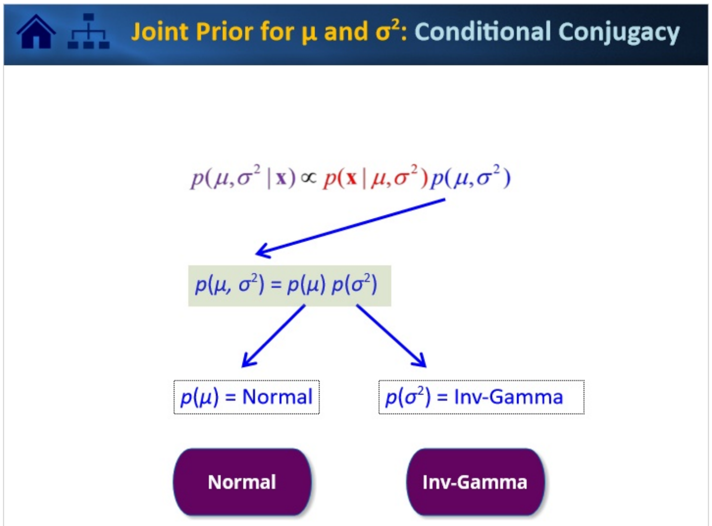
$$p(\mu,\sigma^2 \mid \mathbf{x}) \propto \prod_{i=1}^{n} p(x_i \mid \mu,\sigma^2)\, p(\mu,\sigma^2)$$

$$p(\mathbf{x} \mid \mu,\sigma^2) = \prod_{i=1}^{n} p(x_i \mid \mu,\sigma^2) = \prod_{i=1}^{n} N(x_i \mid \mu,\sigma^2)$$

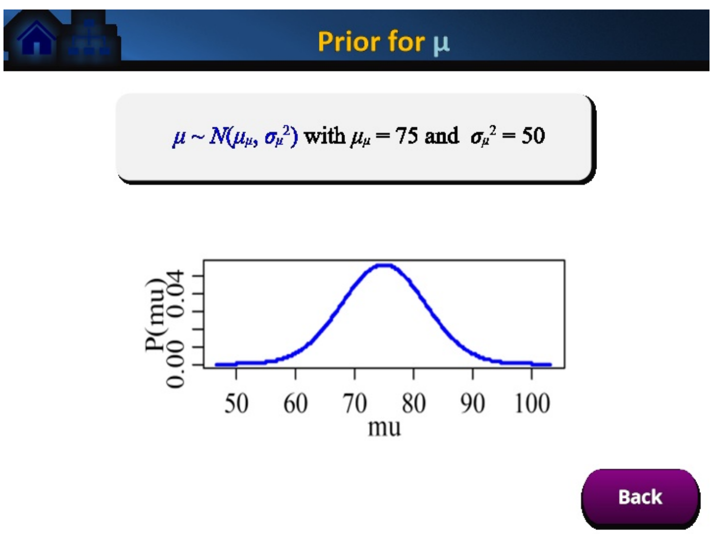
$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left( \frac{1}{2\sigma^2}(x_i - \mu)^2 \right)$$

$$p(\bar{x} \mid \mu,\sigma^2) = N(\bar{x} \mid \mu,\sigma^2/n)$$

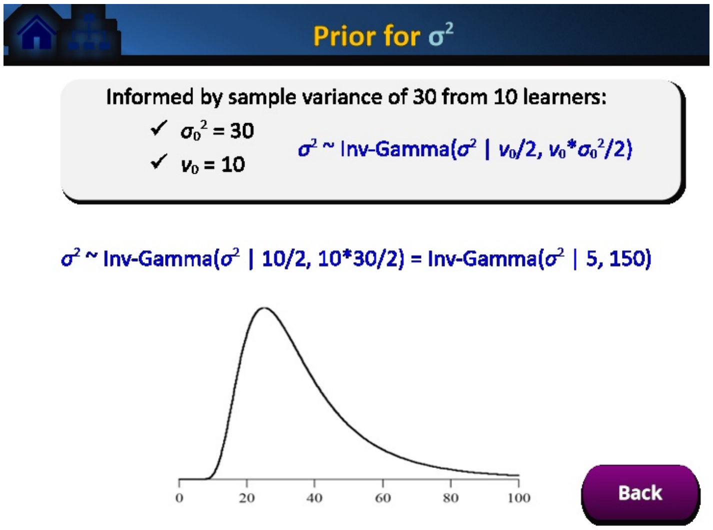
## *3.19 Prior (Mean, Variance)*



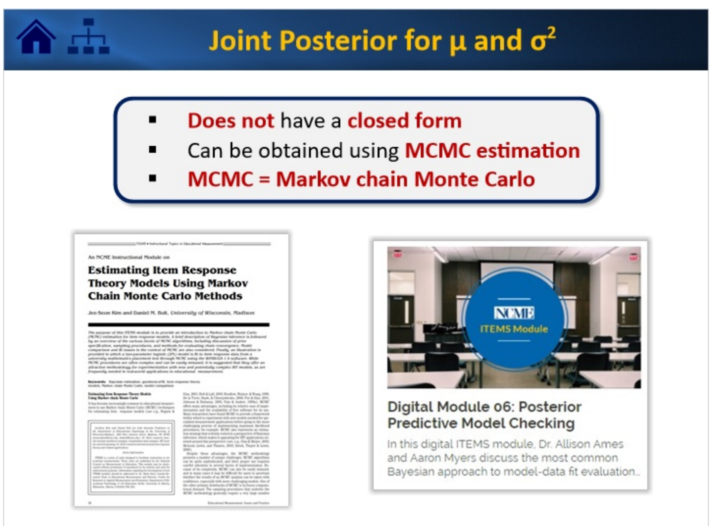

### Prior Mean (Slide Layer)



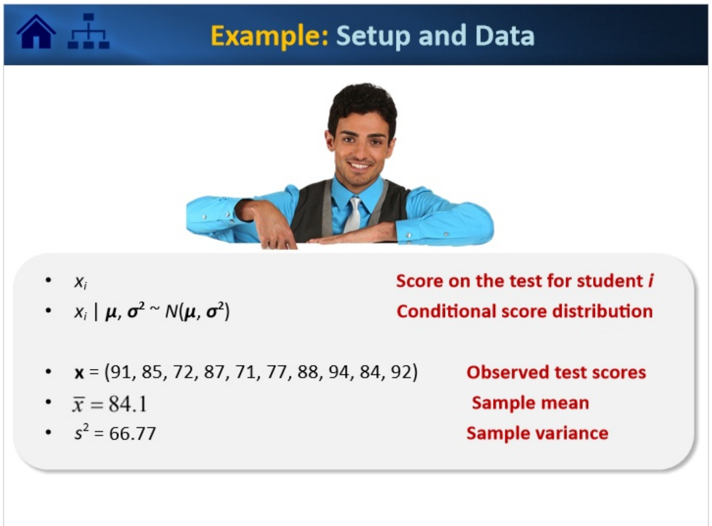
### Prior Variance (Slide Layer)



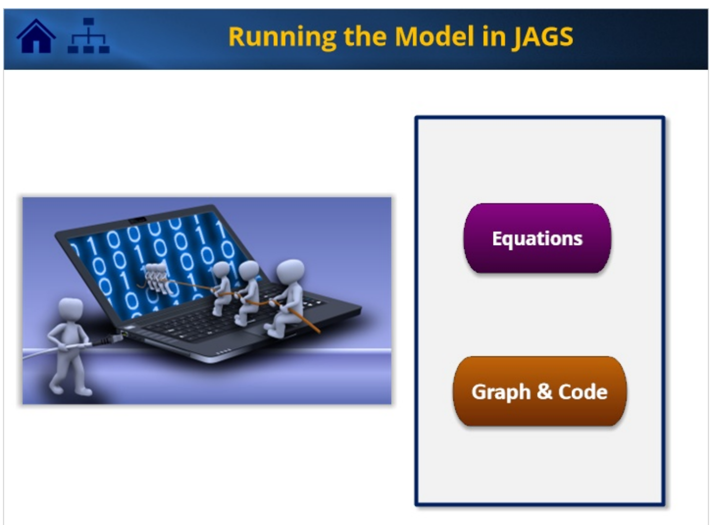

## *3.20 Posterior (Mean, Variance)*



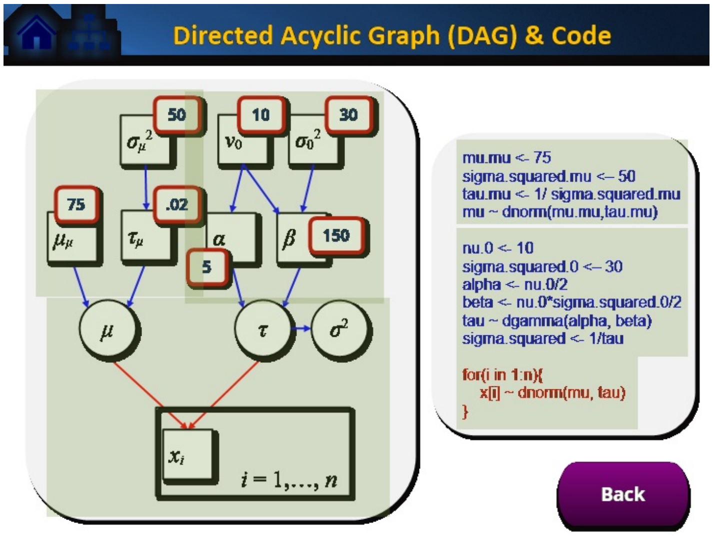
### 3.21 Data Values



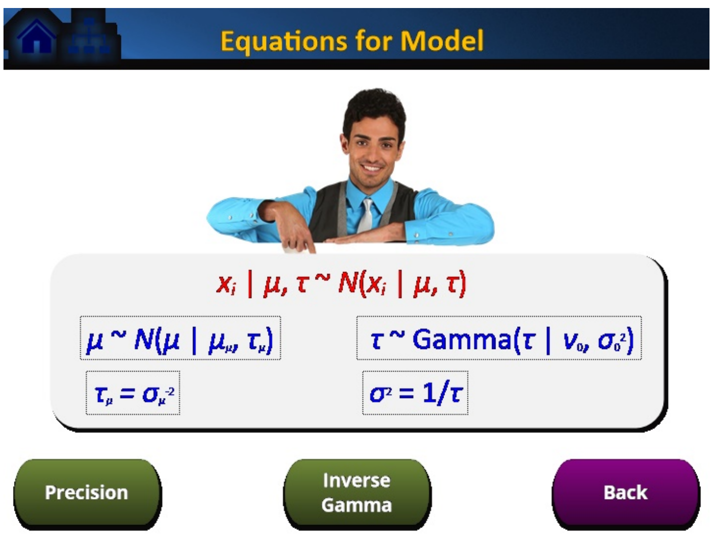

### 3.22 JAGS Implementation



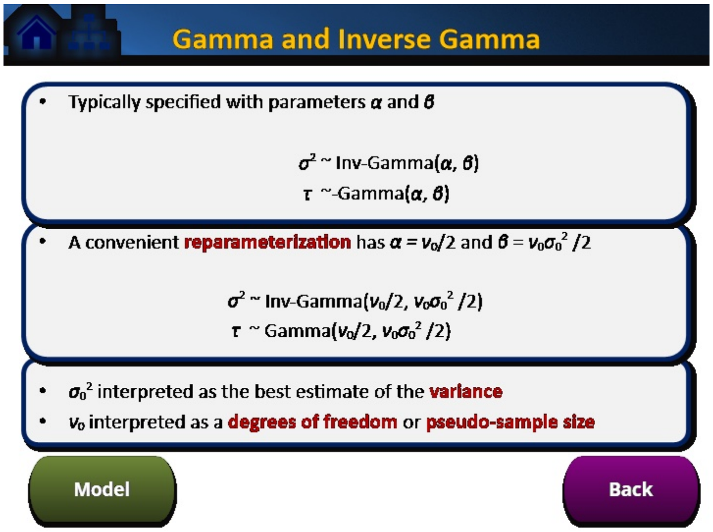
## Graph (Slide Layer)



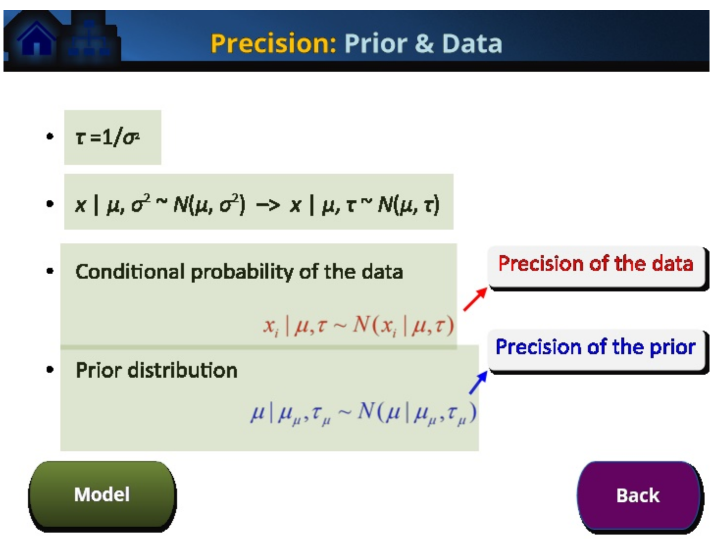

## Equation (Slide Layer)



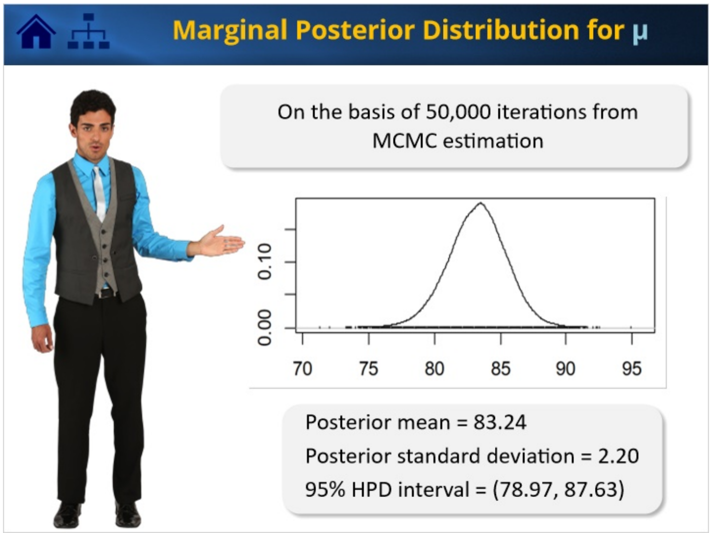
## Gamma (Slide Layer)



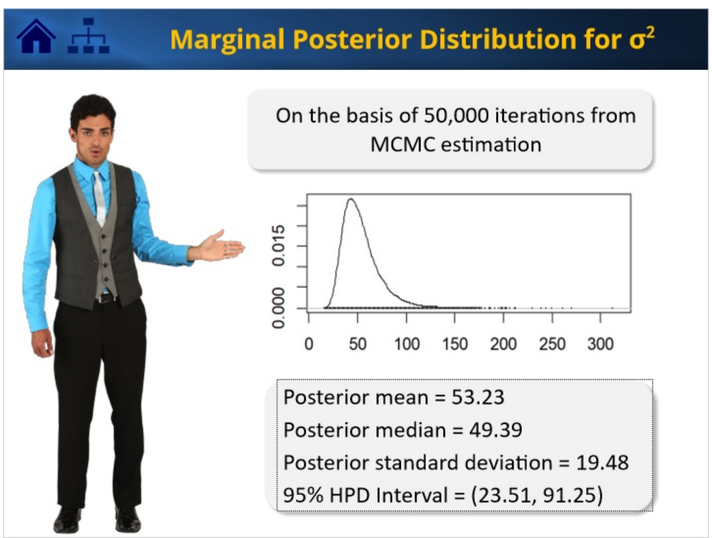

## Precision (I) (Slide Layer)



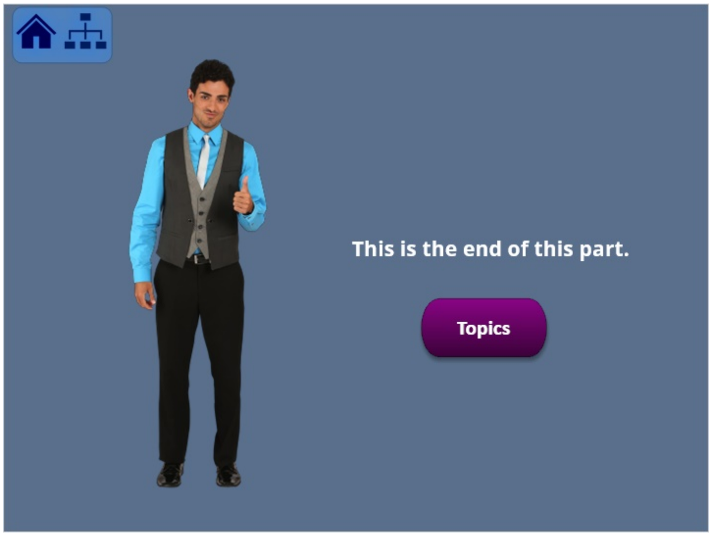
### *3.23 Posterior (Mean)*



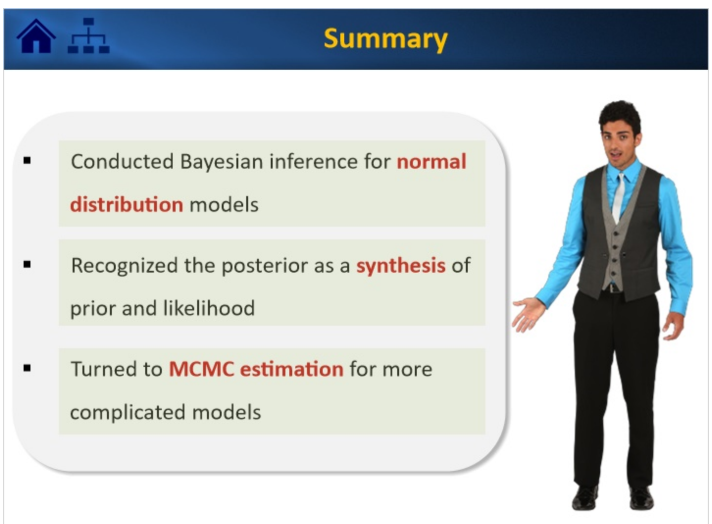

### *3.24 Posterior (Variance)*



## 3.25 Bookend: Setup 2



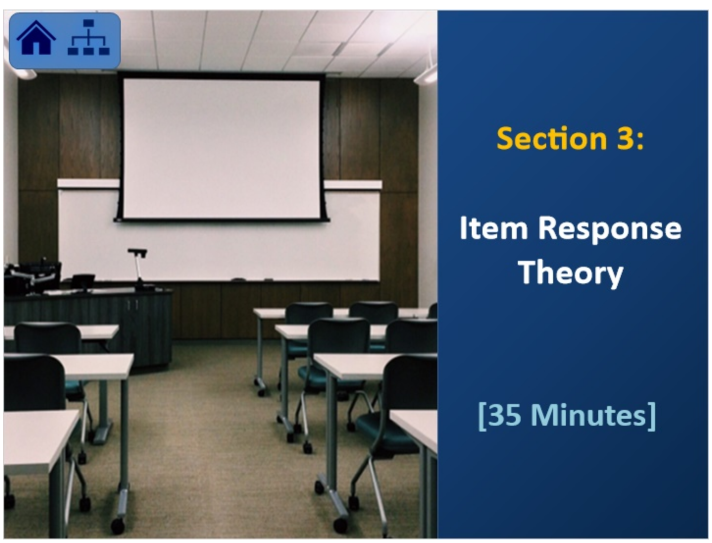

## 3.26 Summary



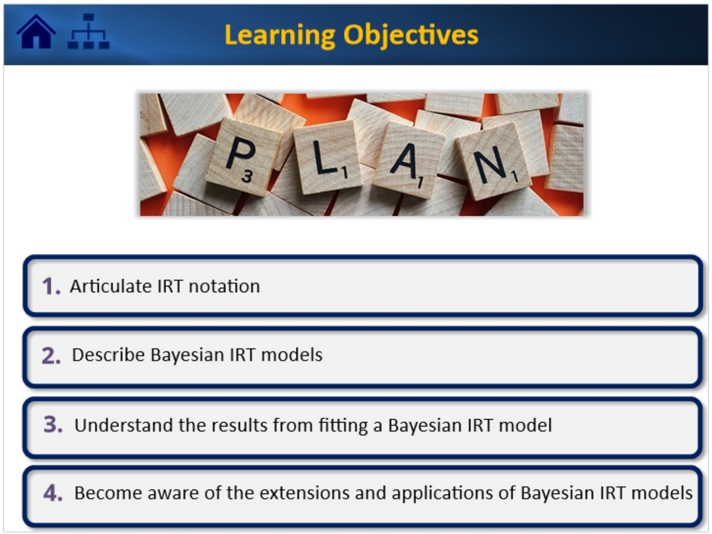
### *3.27 Bookend: Section 2*


## 4. Section 3: Item Response Theory

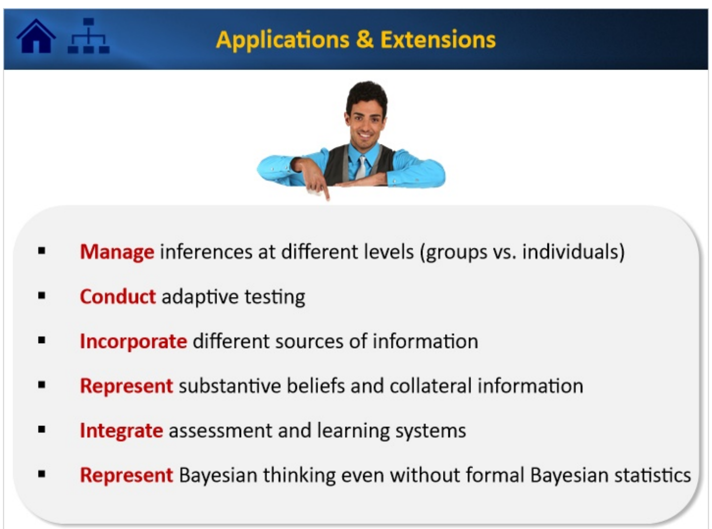
### *4.1 Cover: Section 3  Item Response Theory*



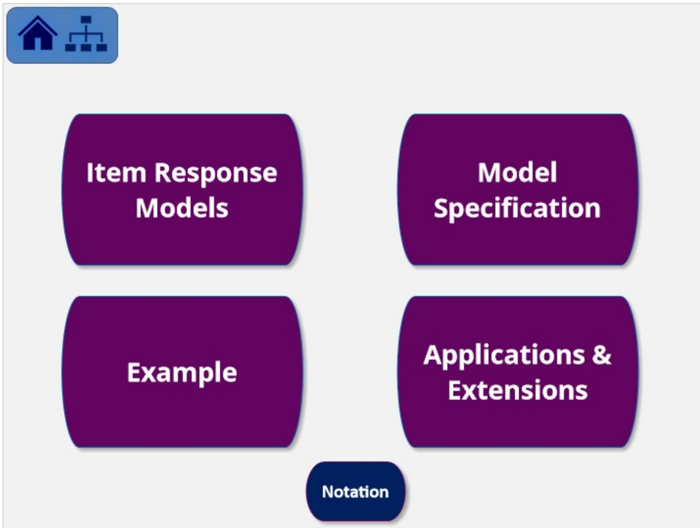
## 4.2 Learning Objectives



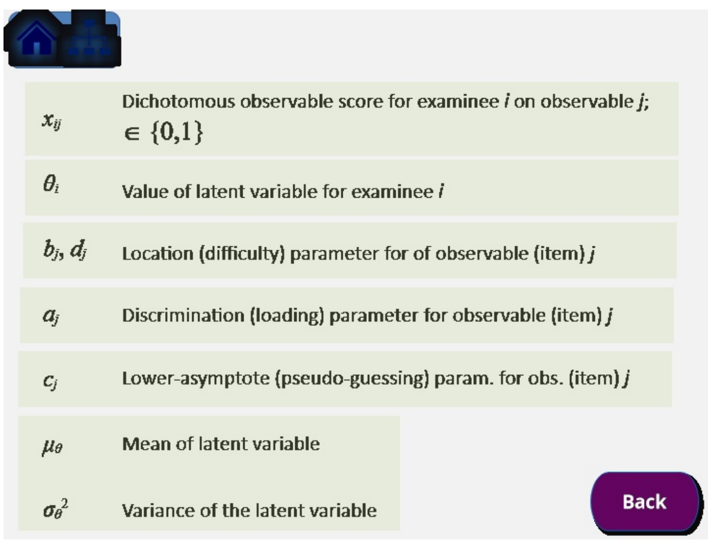

## 4.3 Extensions (II)



## *4.4 Topic Selection*

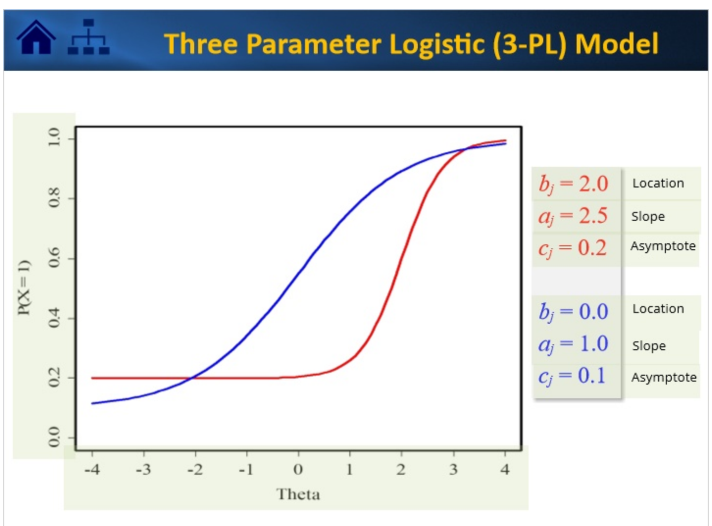



### Notation (Slide Layer)



## *4.5 Bookmark: IRT Models*


## *4.6 Three-parameter Model*



## *4.7 Link Functions*

**Logistic and Normal IRT Models**

**Logistic IRT models** use cumulative ***logistic*** distribution

$$P(x_{ij}=1\,|\,\theta_i,b_j,a_j)=\Psi[a_j(\theta_i-b_j)]=\frac{\exp(a_j(\theta_i-b_j))}{1+\exp(a_j(\theta_i-b_j))}$$

**Normal-Ogive IRT models** use the cumulative ***normal*** distribution

$$P(x_{ij}=1\,|\,\theta_i,b_j,a_j)=\Phi[a_j(\theta_i-b_j)]=\int_{-\infty}^{a_j(\theta_i-b_j)}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right)dt$$

## *4.8 Alternative Parameterizations*

**Alternative Parameterizations**

$$P(x_{ij}=1\,|\,\theta_i,b_j,a_j,c_j)=c_j+(1-c_j)F[a_j(\theta_i-b_j)]$$

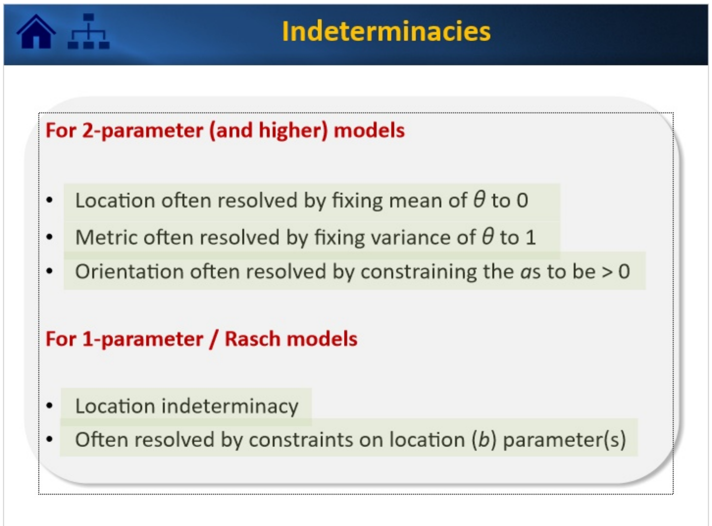
- Focuses on examinee location relative to item
- $P = .5$ when $\theta = b$ for two-parameter models

$$P(x_{ij}=1\,|\,\theta_i,d_j,a_j,c_j)=c_j+(1-c_j)F(a_j\theta_i+d_j)$$

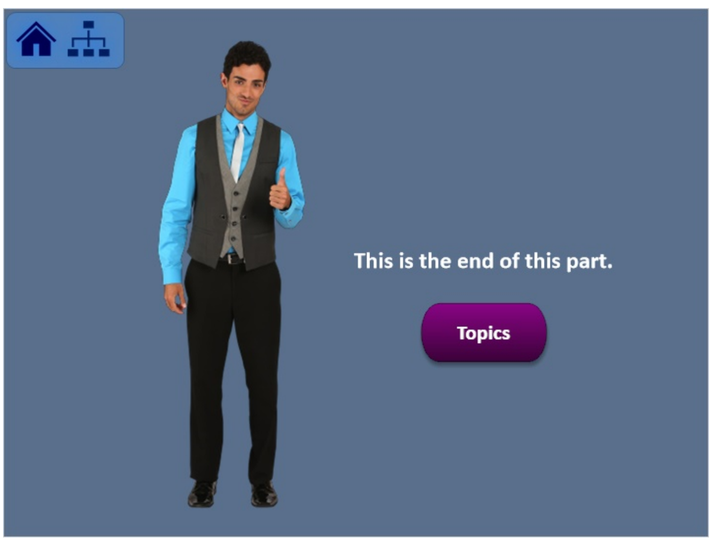
- $d_j = -a_j b_j$
- Useful for viewing IRT similar to regression and factor analysis
- Natural extensions to multidimensional models

***F*** **denotes distribution function (logistic or normal)**

## *4.9 Indeterminacies*

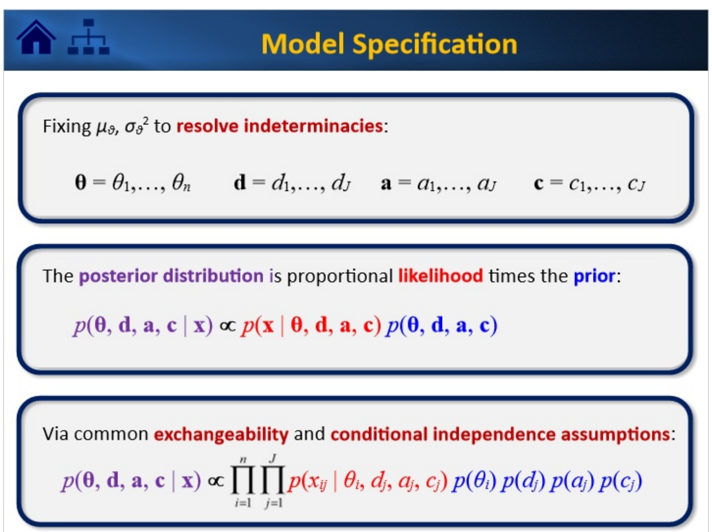



## *4.10 Bookend: IRT Models*

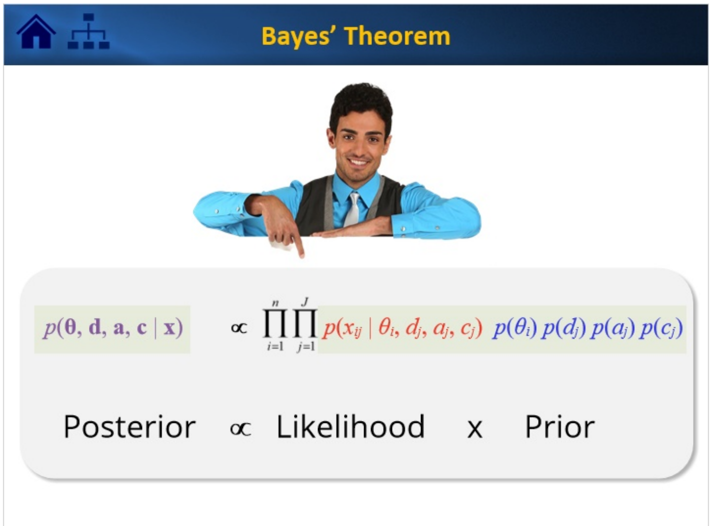


## 4.11 Bookmark: Model Specification

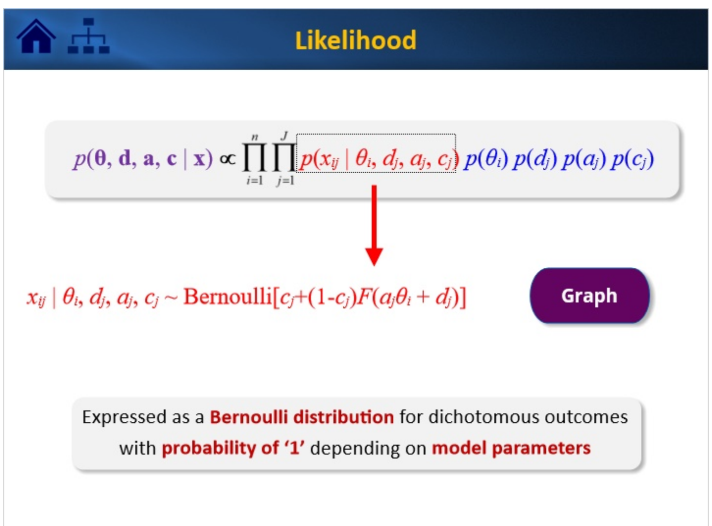

## 4.12 Model Specification



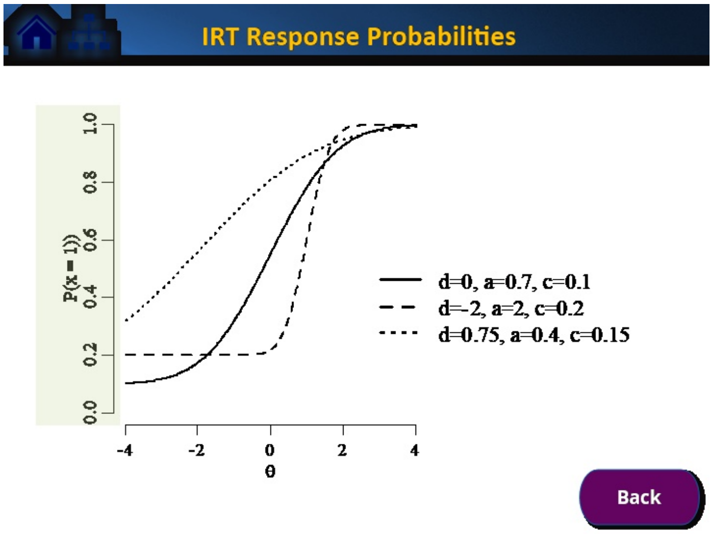
## *4.13 Bayes Theorem*



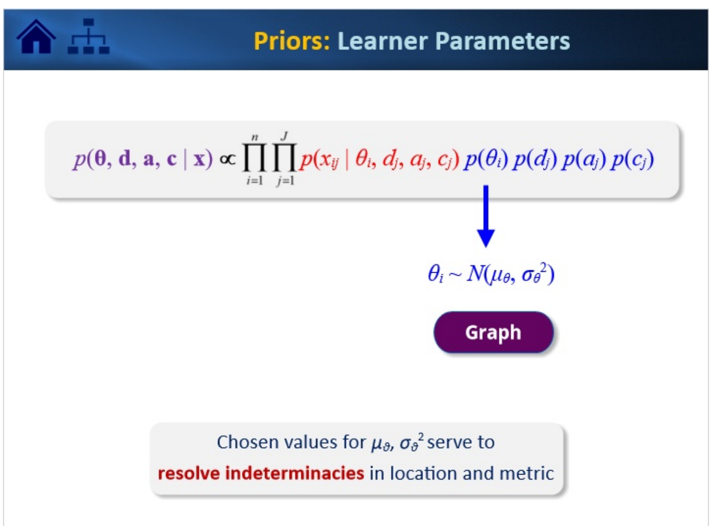

## *4.14 Likelihood*



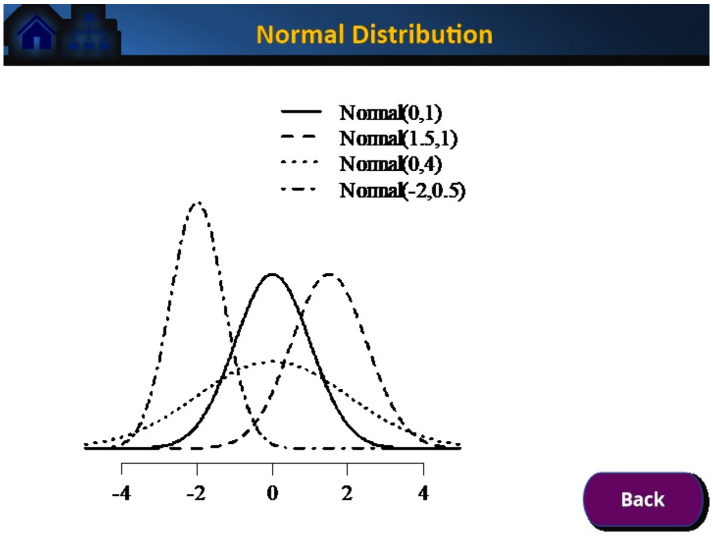
### IRT Probabilities (Slide Layer)



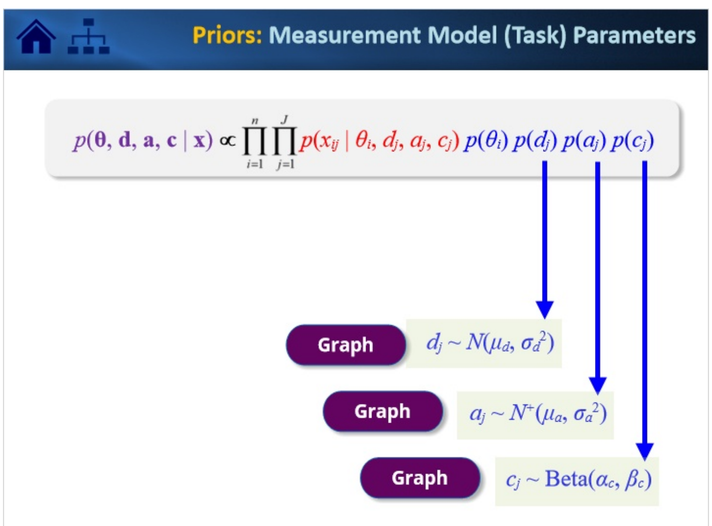

## *4.15 Priors: Learner Parameters*



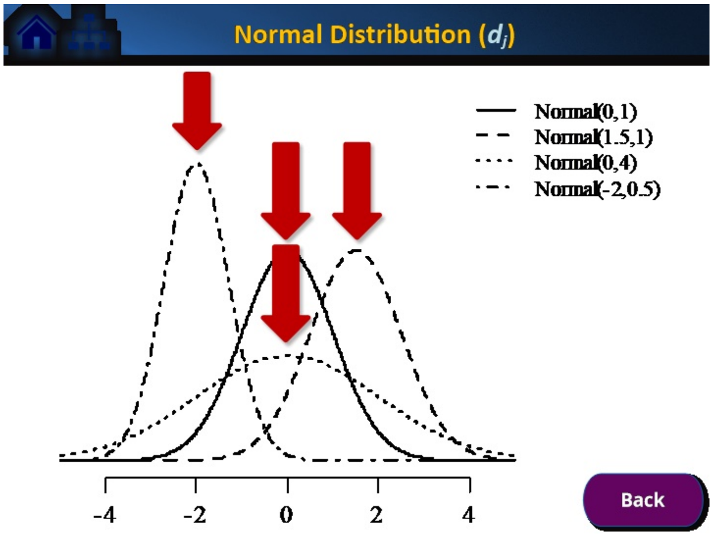
### Graph Normal (Slide Layer)



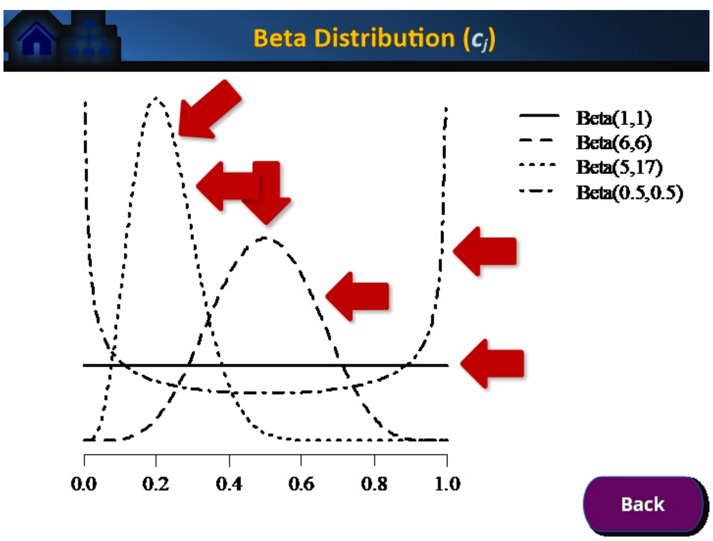

## *4.16 Priors: Item Parameters*



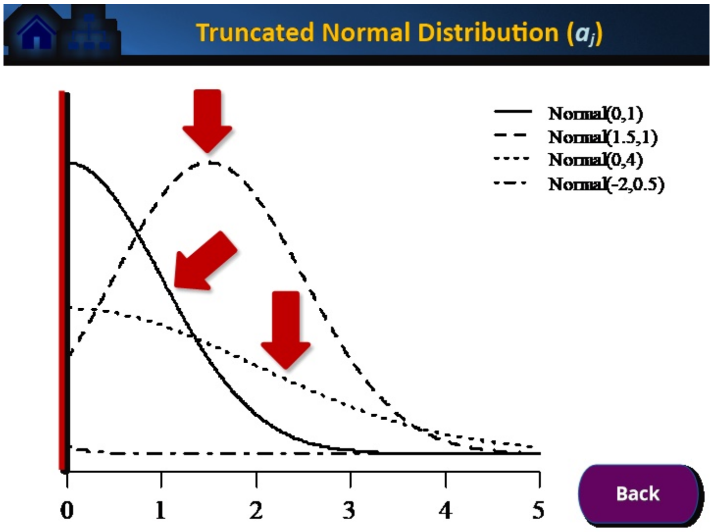
## Graph Normal (Slide Layer)



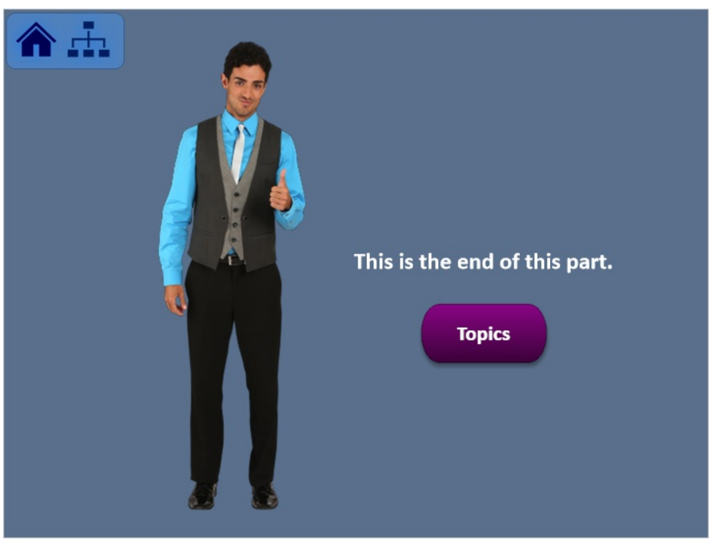

## Graph Beta (Slide Layer)



### Graph Truncated Normal (Slide Layer)




## *4.17 Bookend: Model Specification*



## *4.18 Bookmark: Example*


## *4.19 Example (I)*

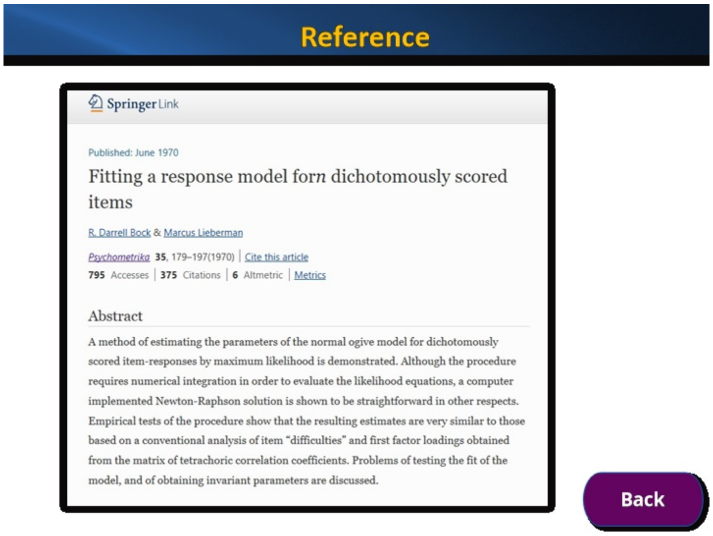
### Example: Law School Admissions Test

| Vector ID | Item 1 | 2 | 3 | 4 | 5 | Frequency | Vector ID | Item 1 | 2 | 3 | 4 | 5 | Frequency |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 3 | 17 | 1 | 0 | 0 | 0 | 0 | 10 |
| 2 | 0 | 0 | 0 | 0 | 1 | 6 | 18 | 1 | 0 | 0 | 0 | 1 | 29 |
| 3 | 0 | 0 | 0 | 1 | 0 | 2 | 19 | 1 | 0 | 0 | 1 | 0 | 14 |
| 4 | 0 | 0 | 0 | 1 | 1 | 11 | 20 | 1 | 0 | 0 | 1 | 1 | 81 |
| 5 | 0 | 0 | 1 | 0 | 0 | 1 | 21 | 1 | 0 | 1 | 0 | 0 | 3 |
| 6 | 0 | 0 | 1 | 0 | 1 | 1 | 22 | 1 | 0 | 1 | 0 | 1 | 28 |
| 7 | 0 | 0 | 1 | 1 | 0 | 3 | 23 | 1 | 0 | 1 | 1 | 0 | 15 |
| 8 | 0 | 0 | 1 | 1 | 1 | 4 | 24 | 1 | 0 | 1 | 1 | 1 | 80 |
| 9 | 0 | 1 | 0 | 0 | 0 | 1 | 25 | 1 | 1 | 0 | 0 | 0 | 16 |
| 10 | 0 | 1 | 0 | 0 | 1 | 8 | 26 | 1 | 1 | 0 | 0 | 1 | 56 |
| 11 | 0 | 1 | 0 | 1 | 0 | 0 | 27 | 1 | 1 | 0 | 1 | 0 | 21 |
| 12 | 0 | 1 | 0 | 1 | 1 | 16 | 28 | 1 | 1 | 0 | 1 | 1 | 173 |
| 13 | 0 | 1 | 1 | 0 | 0 | 0 | 29 | 1 | 1 | 1 | 0 | 0 | 11 |
| 14 | 0 | 1 | 1 | 0 | 1 | 3 | 30 | 1 | 1 | 1 | 0 | 1 | 61 |
| 15 | 0 | 1 | 1 | 1 | 0 | 2 | 31 | 1 | 1 | 1 | 1 | 0 | 28 |
| 16 | 0 | 1 | 1 | 1 | 1 | 15 | 32 | 1 | 1 | 1 | 1 | 1 | 298 |

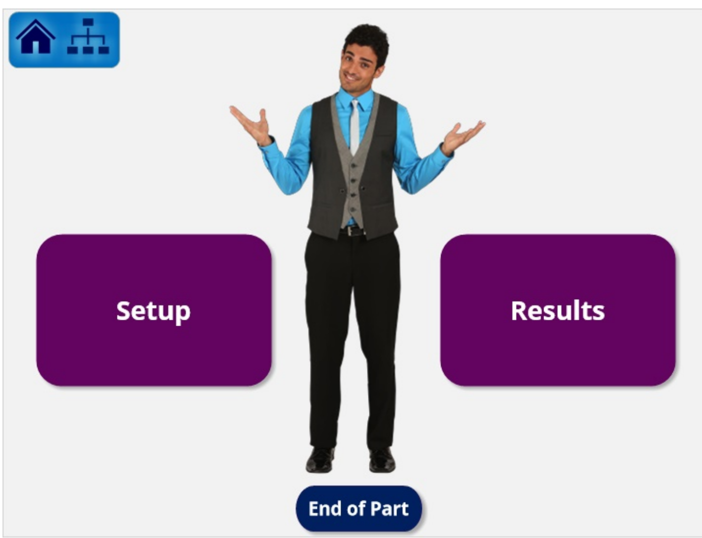
Summary of Response Patterns (5 items, 1000 examinees)

Reference

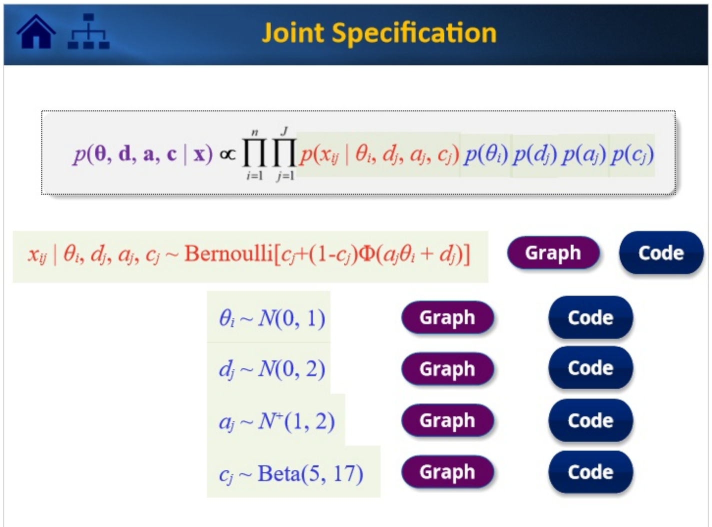
## Reference (Slide Layer)



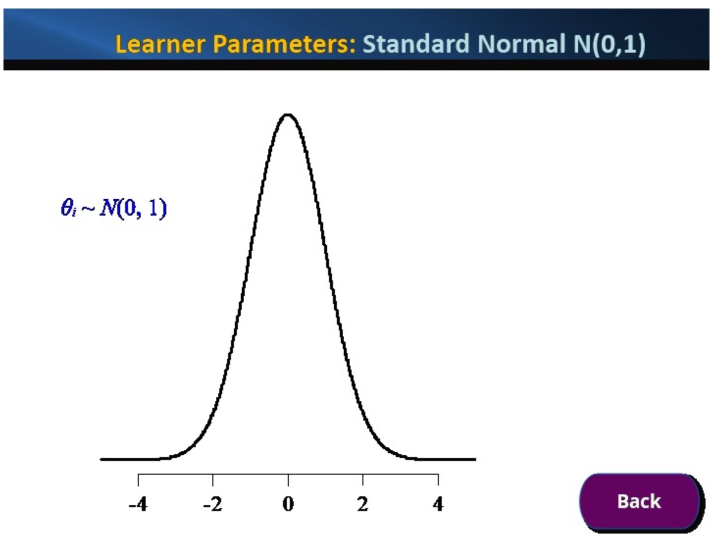

## *4.20 Topic Selection*



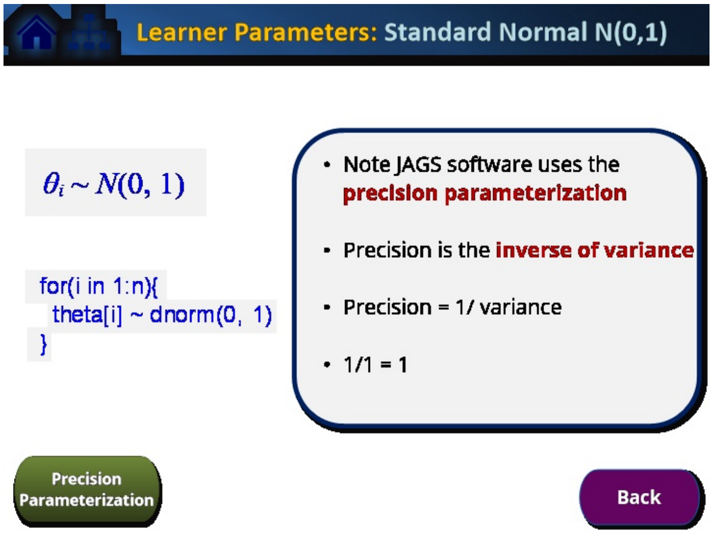
## *4.21 Example (II)*



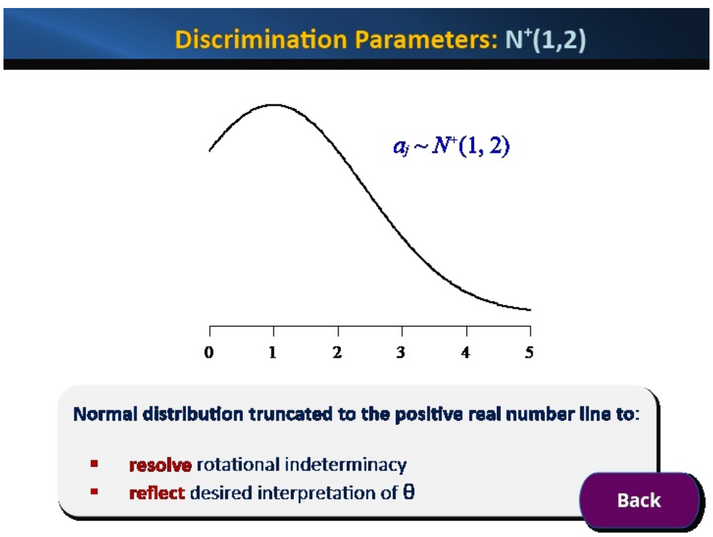

### Theta Graph (Slide Layer)



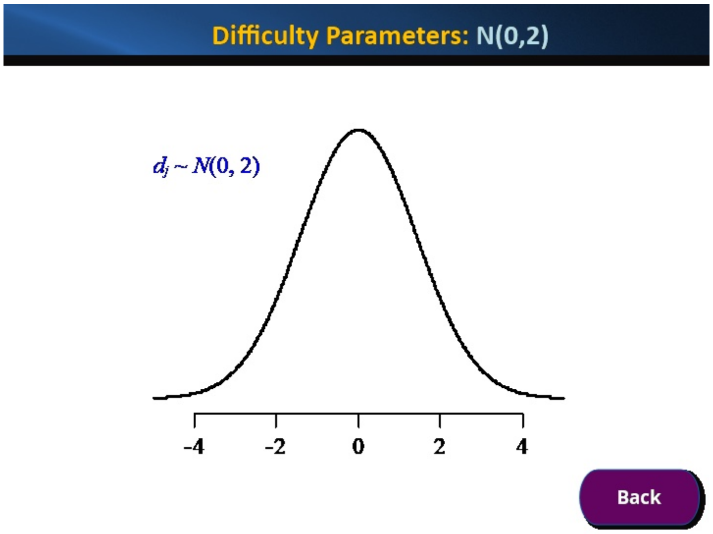
## Theta Code (Slide Layer)



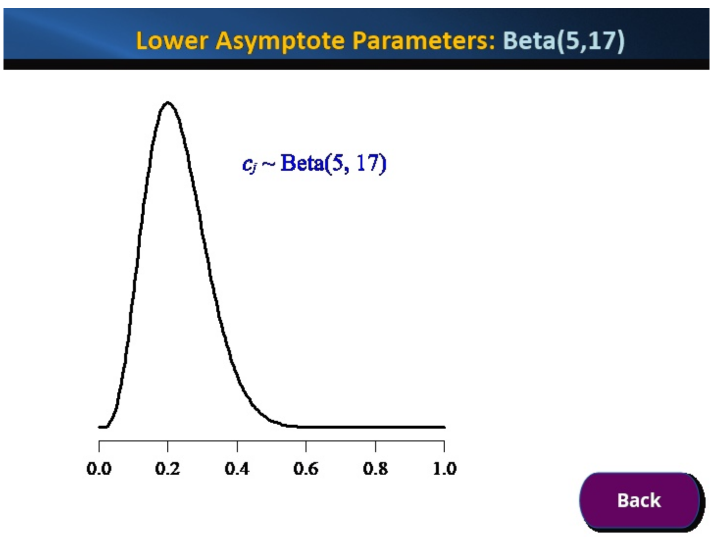

## Discrimination Graph (Slide Layer)



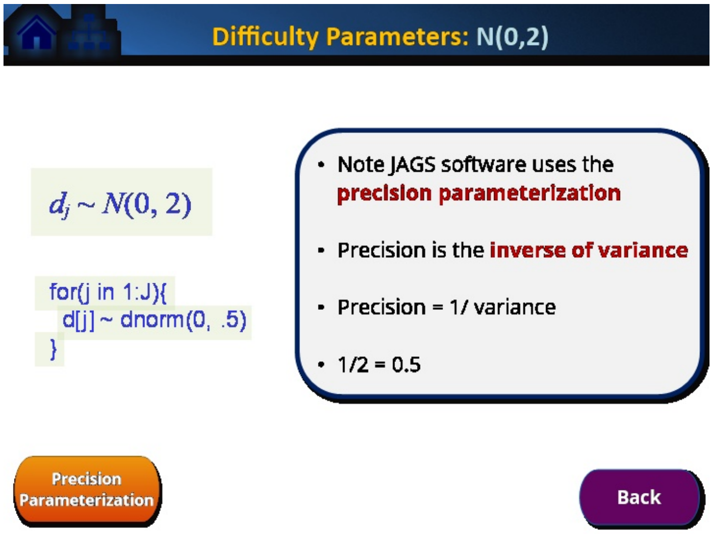
## Difficulty Graph (Slide Layer)



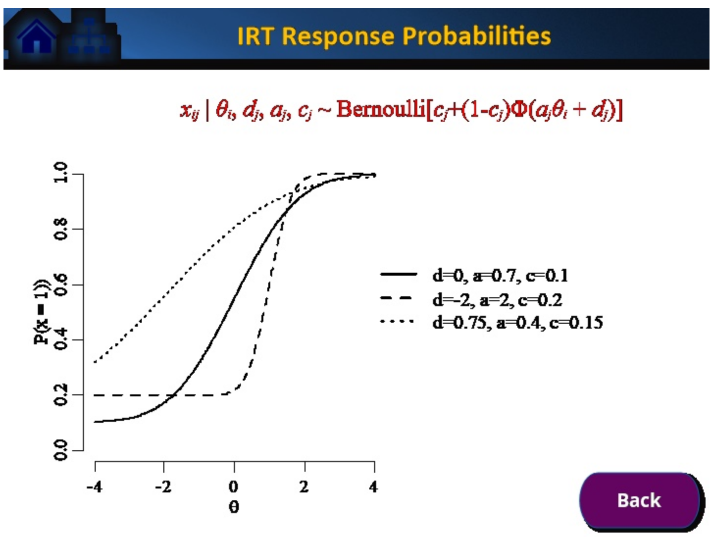

## Guessing Graph (Slide Layer)



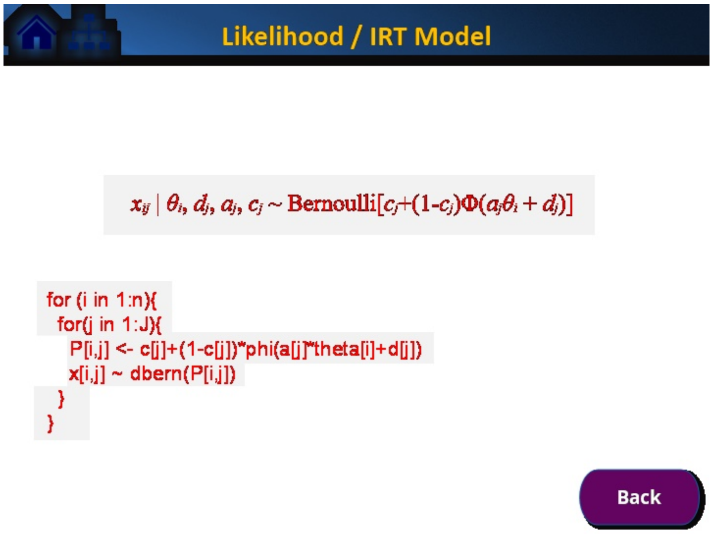
## Difficulty Code (Slide Layer)



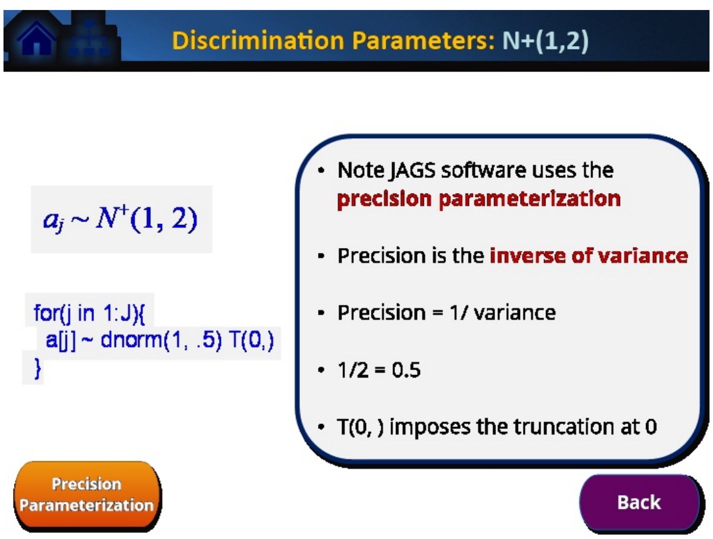

## IRT Curves Graph (Slide Layer)



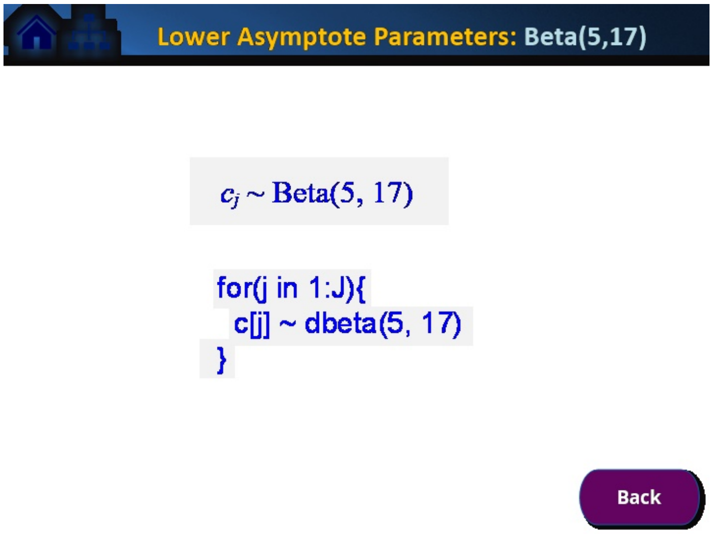
## IRT Model Code (Slide Layer)



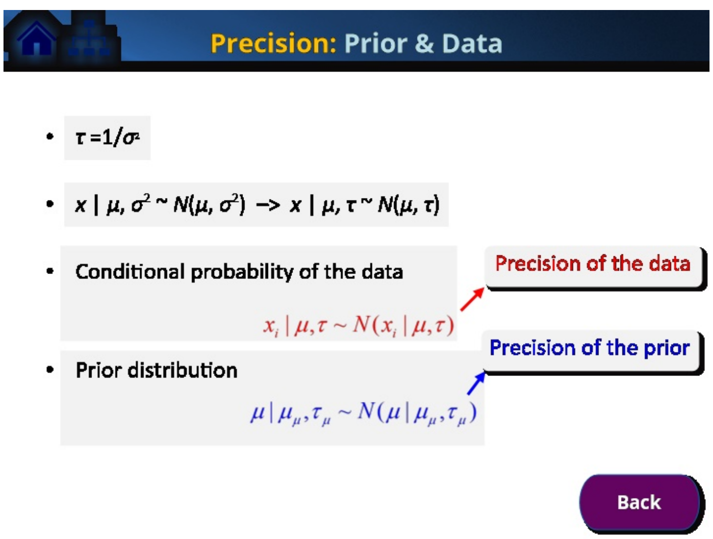

## Discrimination Code (Slide Layer)



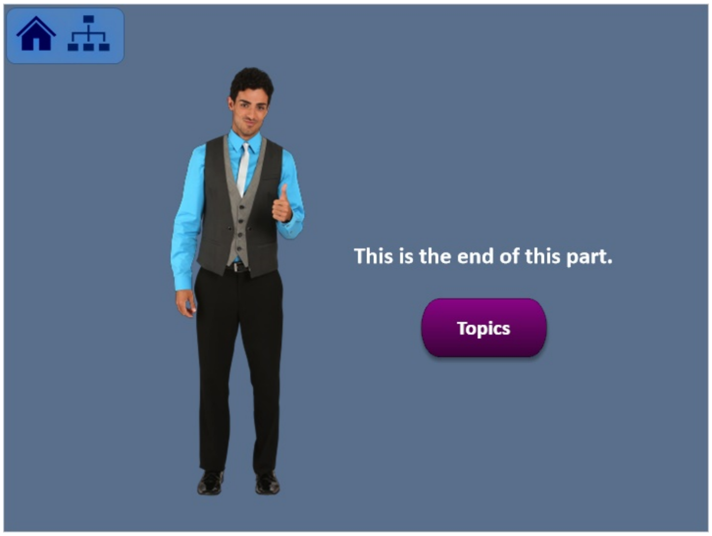
## Guessing Code (Slide Layer)



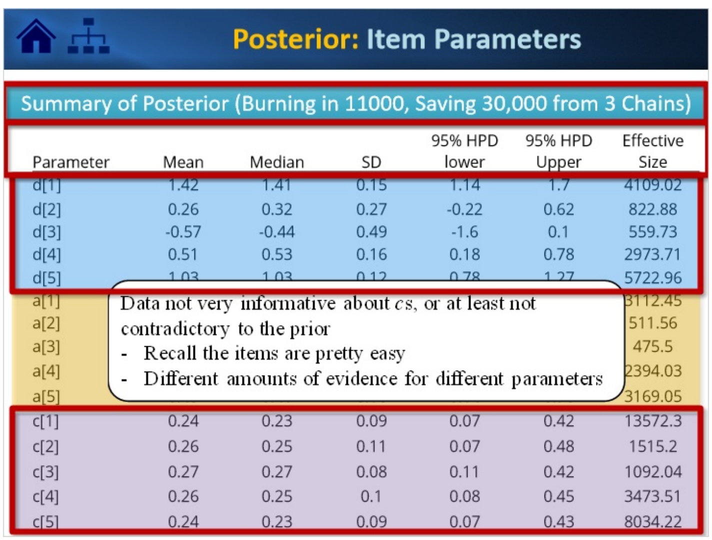

## Precision (Slide Layer)



## *4.22 Bookend: Setup*




## *4.23 Posterior: Item Parameters*




| Parameter | Mean | Median | SD | 95% HPD lower | 95% HPD Upper | Effective Size |
|---|---|---|---|---|---|---|
| d[1] | 1.42 | 1.41 | 0.15 | 1.14 | 1.7 | 4109.02 |
| d[2] | 0.26 | 0.32 | 0.27 | -0.22 | 0.62 | 822.88 |
| d[3] | -0.57 | -0.44 | 0.49 | -1.6 | 0.1 | 559.73 |
| d[4] | 0.51 | 0.53 | 0.16 | 0.18 | 0.78 | 2973.71 |
| d[5] | 1.03 | 1.03 | 0.12 | 0.78 | 1.27 | 5722.96 |
| a[1] | | | | | | 3112.45 |
| a[2] | | | | | | 511.56 |
| a[3] | | | | | | 475.5 |
| a[4] | | | | | | 2394.03 |
| a[5] | | | | | | 3169.05 |
| c[1] | 0.24 | 0.23 | 0.09 | 0.07 | 0.42 | 13572.3 |
| c[2] | 0.26 | 0.25 | 0.11 | 0.07 | 0.48 | 1515.2 |
| c[3] | 0.27 | 0.27 | 0.08 | 0.11 | 0.42 | 1092.04 |
| c[4] | 0.26 | 0.25 | 0.1 | 0.08 | 0.45 | 3473.51 |
| c[5] | 0.24 | 0.23 | 0.09 | 0.07 | 0.43 | 8034.22 |

## *4.24 Extensions (I)*

**Extensions**

- Complex models
- Polyomtous observables
- Multiple latent variables
- Alternative priors
- Hierarchical specifications
- Covariates
- Other modeling families

## *4.25 Posterior (Learners)*

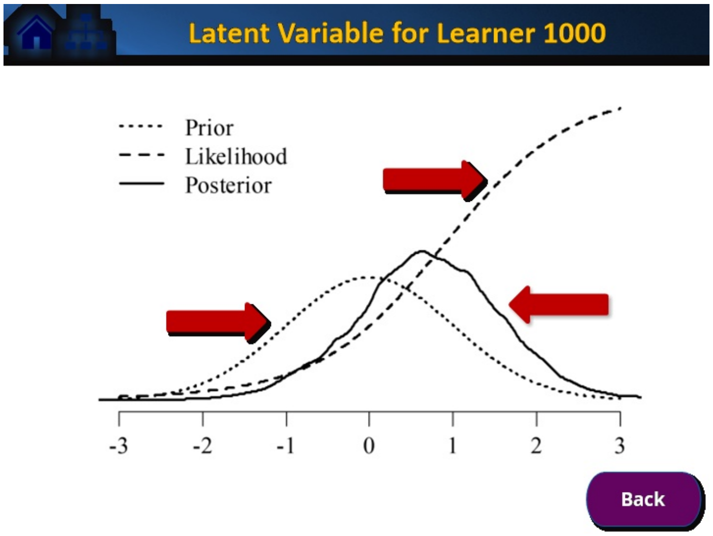
**Posterior: Learner Parameters**

Summary of Posterior (Burning in 11000, Saving 30,000 from 3 Chains)

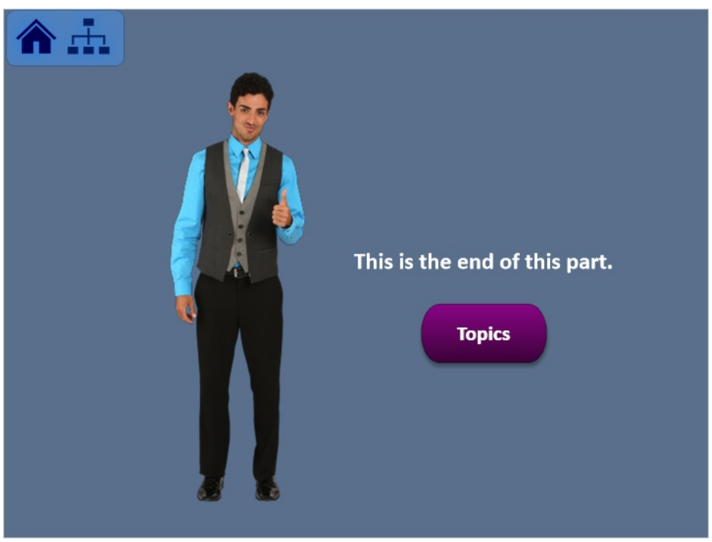
| Parameter | Mean | Median | SD | 95% HPD lower | 95% HPD Upper | Effective Size |
|---|---|---|---|---|---|---|
| theta[1] | -1.63 | -1.61 | 0.77 | -3.18 | -0.17 | 50473.72 |
| . | . | . | . | . | . | . |
| . | . | . | . | . | . | . |
| . | . | . | . | . | . | . |
| theta[1000] | 0.70 | 0.72 | 0.88 | -1.10 | 2.38 | 54701.95 |

**Example**

**Example (Slide Layer)**

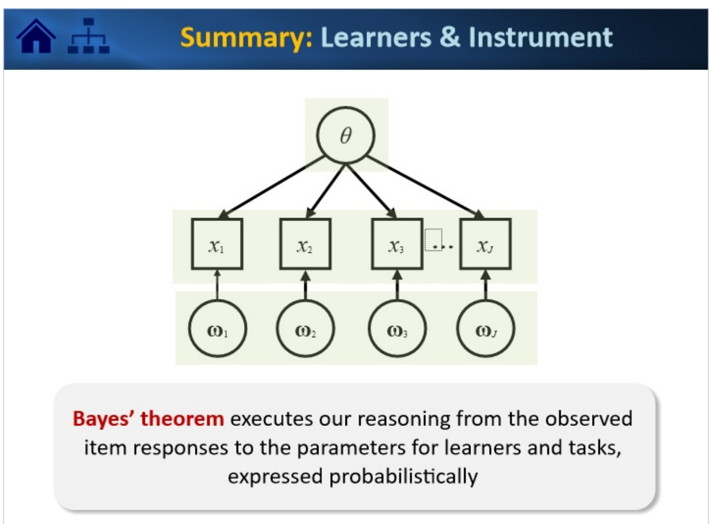



### *4.26 Bookend: Example*



## 4.27 Bookmark: Applications and Extensions


## 4.28 Summary



## 4.29 Advantages


## 4.30 Bookend: Section 3