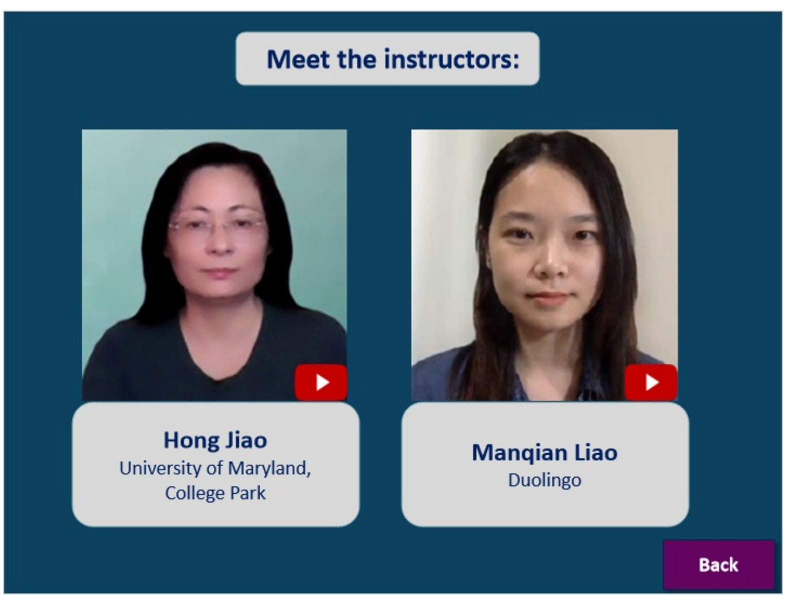
DM25 SLIDES (Testlet Models, Version 1.0)

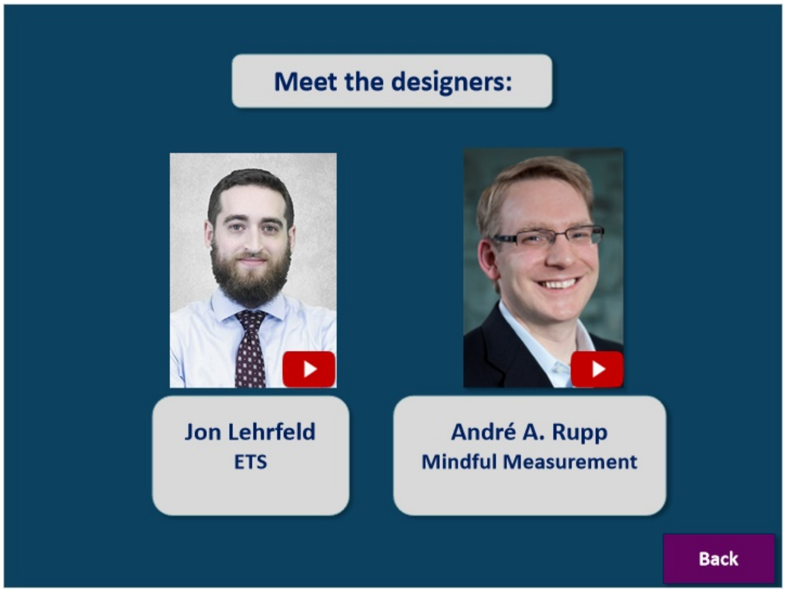
1. Module Overview

1.1 Module Cover (START)

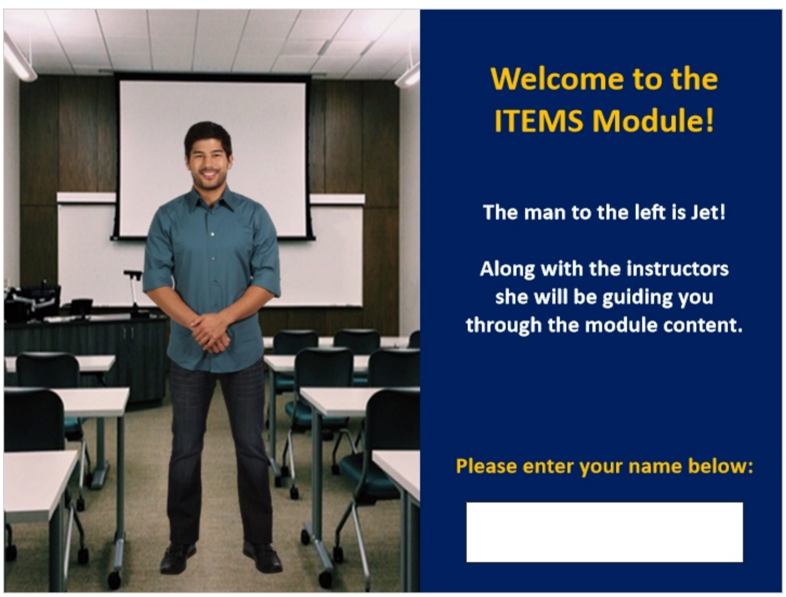
1.2 Instructors



1.3 Designers



1.4 Welcome



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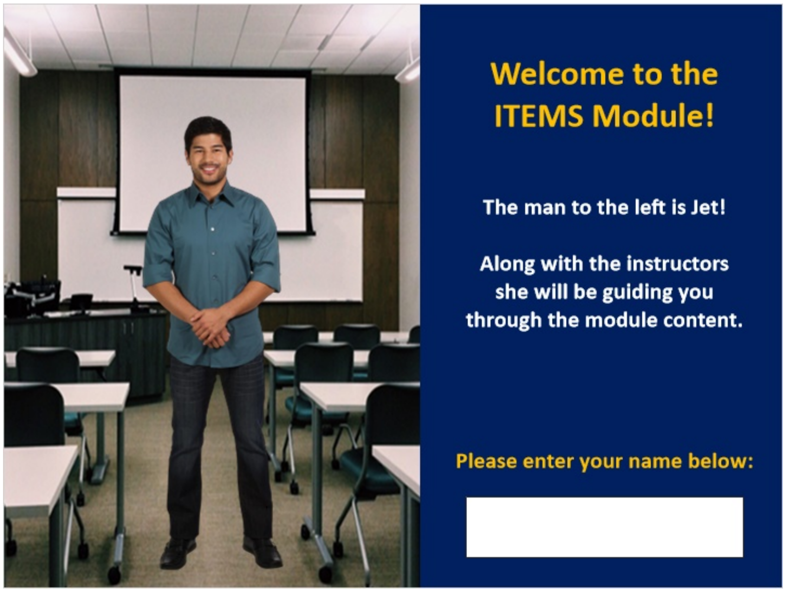
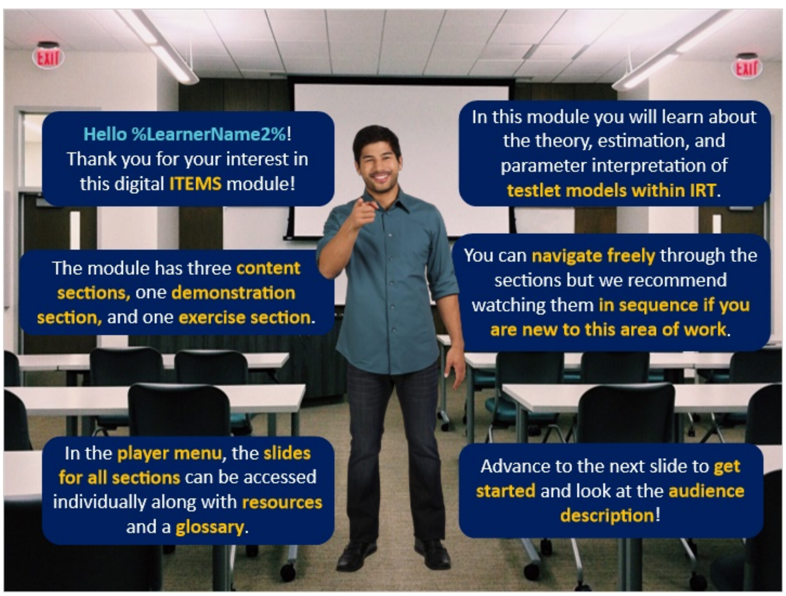
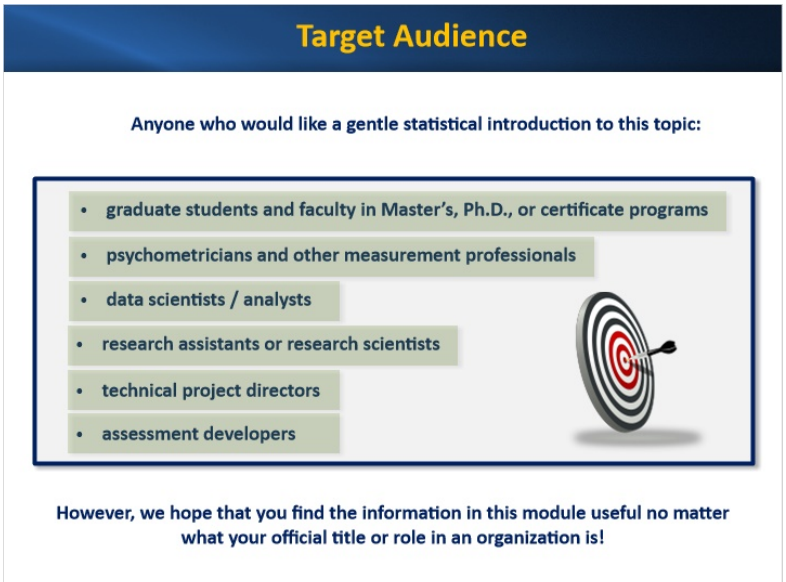


1.5 Overview



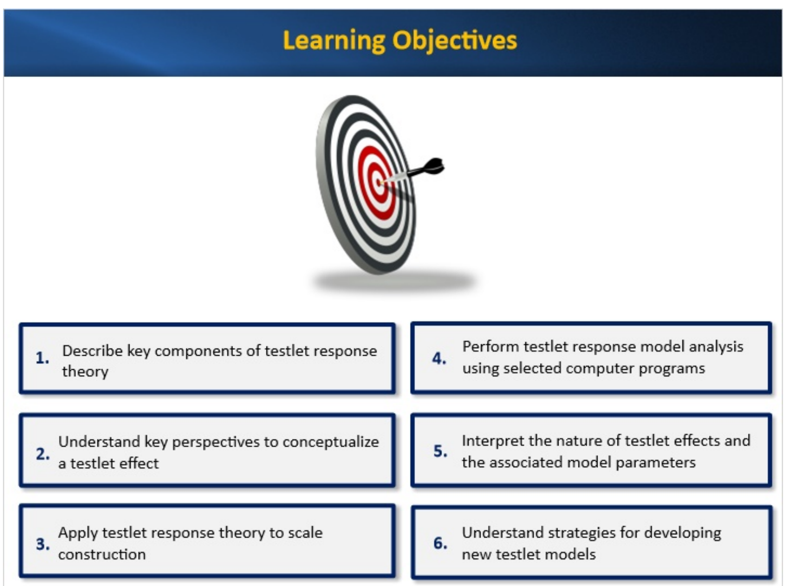
1.6 Target Audience



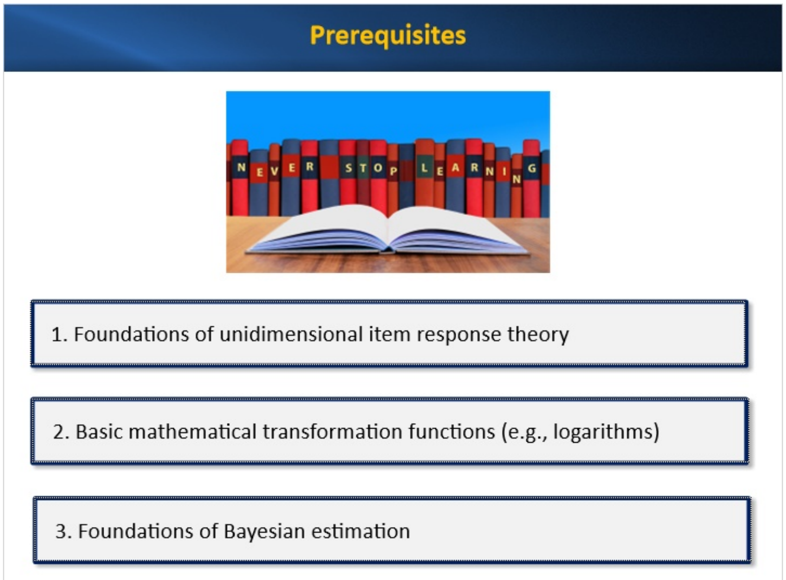
1.7 Expecations (I)

1.8 Expectations (II)

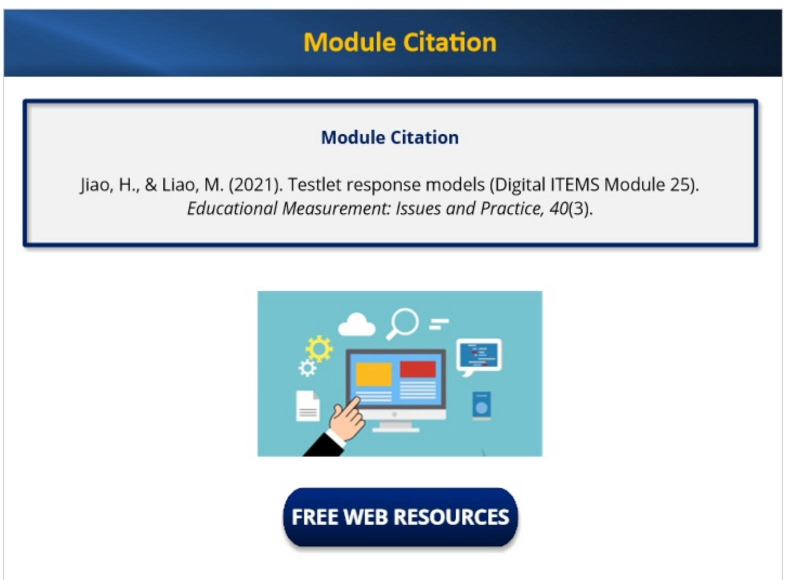
1.9 Learning Objectives



1.10 Prerequisites



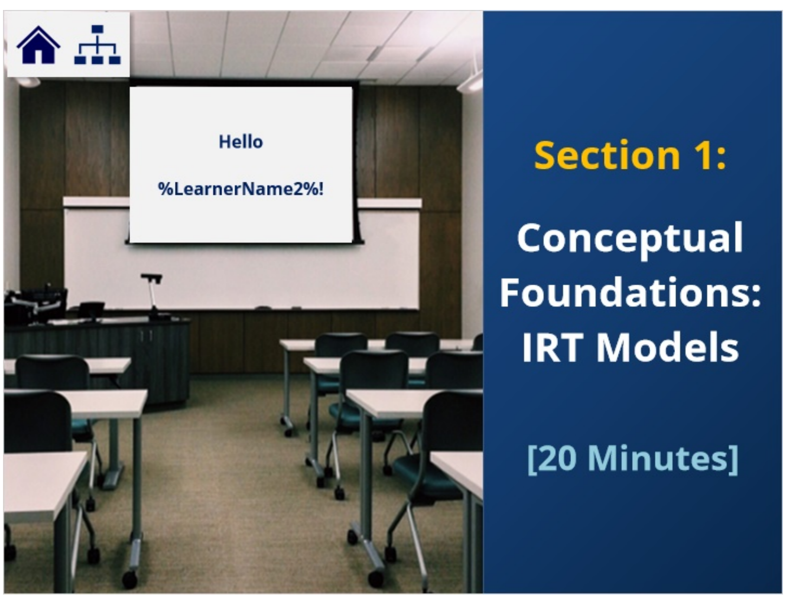
1.11 Module Citation



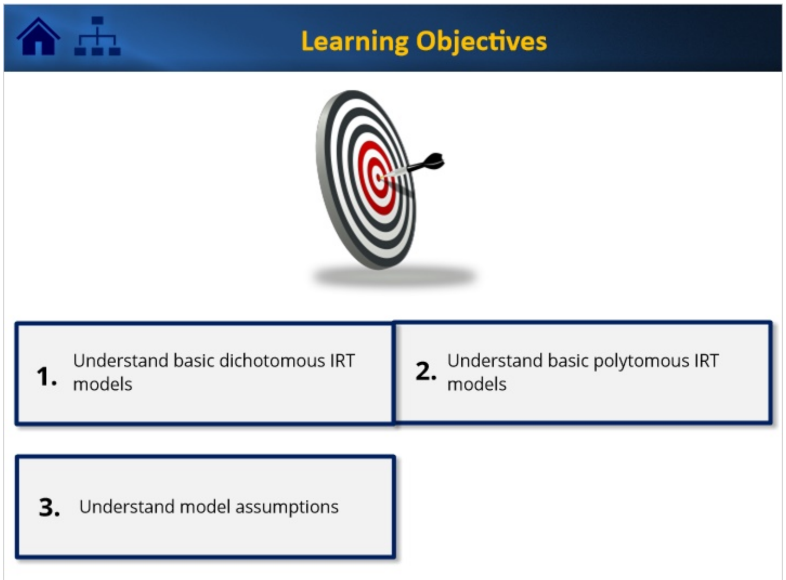
1.12 Main Menu

2. Section 1: Conceptual Foundations: IRT Models

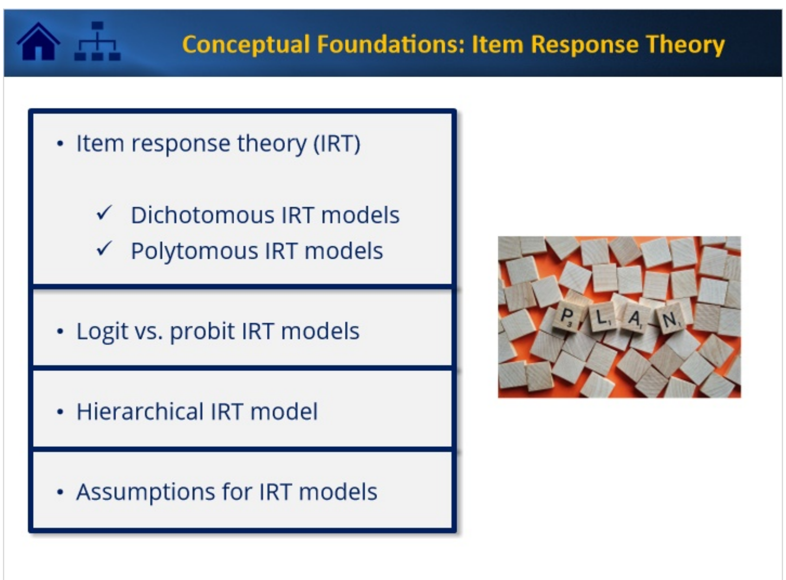
2.1 Cover: Section 1



2.2 Learning Objectives: Section 1



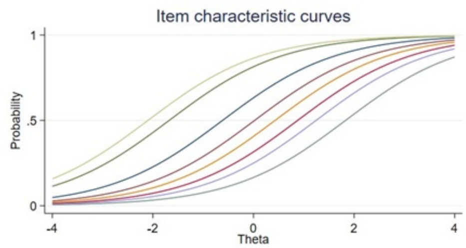
2.3 Conceptual Foundations-Item Response Theory



2.4 Item Response Theory

Item Response Theory (IRT)

- A modern measurement framework that quantifies the relationship among an item response and a person's latent ability given item characteristics.



adapted from https://www.stata.com/features/overview/irt/

- A mathematical representation modeling the nonlinear relationship between the probability of an item response, a latent ability level, and item parameters.

2.5 Item Response Theory Models

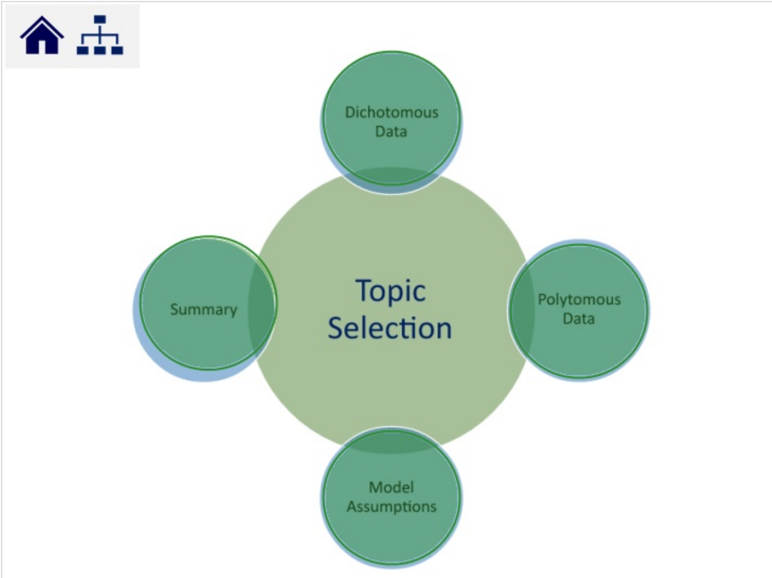
IRT Models

- The common IRT models include models for dichotomous responses and polytomous responses.

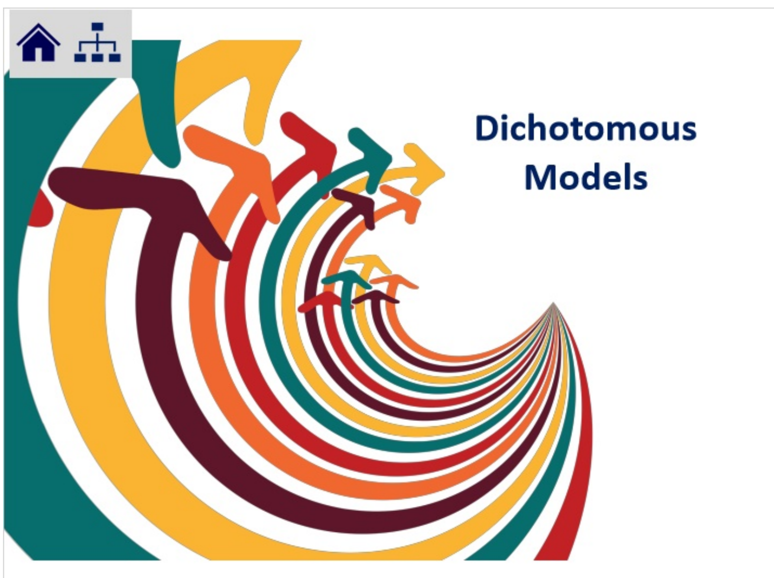
- Common IRT models for dichotomous item responses:
  - ✓ the Rasch measurement model
  - ✓ the two-parameter IRT model
  - ✓ the three-parameter IRT model

- Common IRT models for polytomous item responses:
  - ✓ the partial credit model
  - ✓ the generalized partial credit model
  - ✓ the graded response model

2.6 Topic Selection



2.7 Bookmark: Dichotomous Models



2.8 The Rasch Measurement Model I

The Rasch Measurement Model

- A **latent logistic model** modeling the **nonlinear relationship** between the probability of an item response and a latent ability level given an item difficulty parameter

- Such a nonlinear relationship can be graphically represented in an **item characteristic curve** (ICC) to show how an increase in latent ability leads to an increase in the probability of a correct item response given the specific item difficulty

- The **logit of a correct response** is linearly related to the interaction between a person's latent ability, ***θ***, and the item difficulty, *b*

2.9 The Rasch Measurement Model II

The Rasch Measurement Model

Logistic model

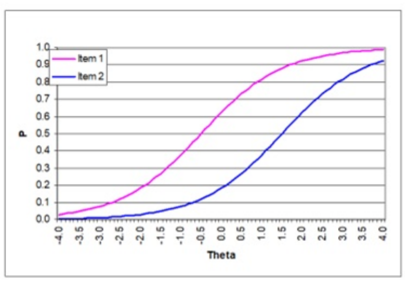
$$p_{ij}(x_{ij}=1|\theta_j)=\frac{\exp(\theta_j-b_i)}{1+\exp(\theta_j-b_i)}$$

$$p_{ij}(x_{ij}=1|\theta_j)=\frac{1}{1+\exp[-(\theta_j-b_i)]}$$

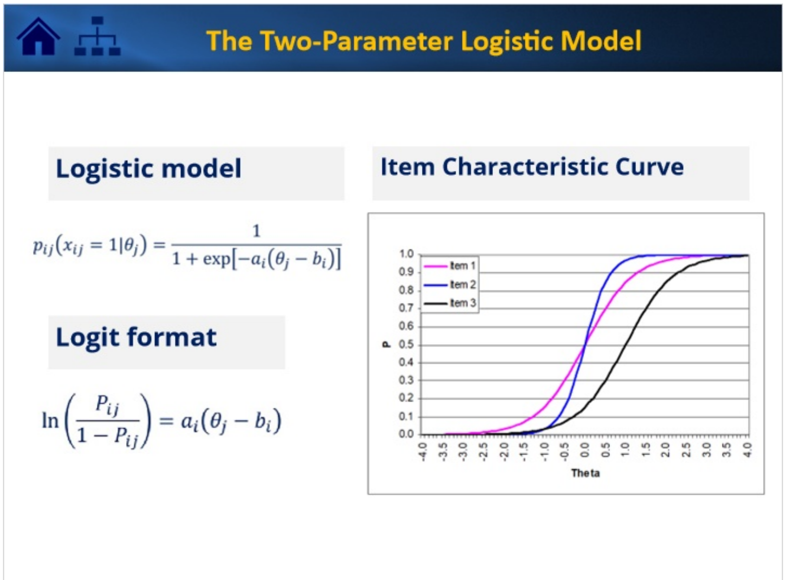
Logit format

$$\ln\left(\frac{P_{ij}}{1-P_{ij}}\right)=(\theta_j-b_i)$$

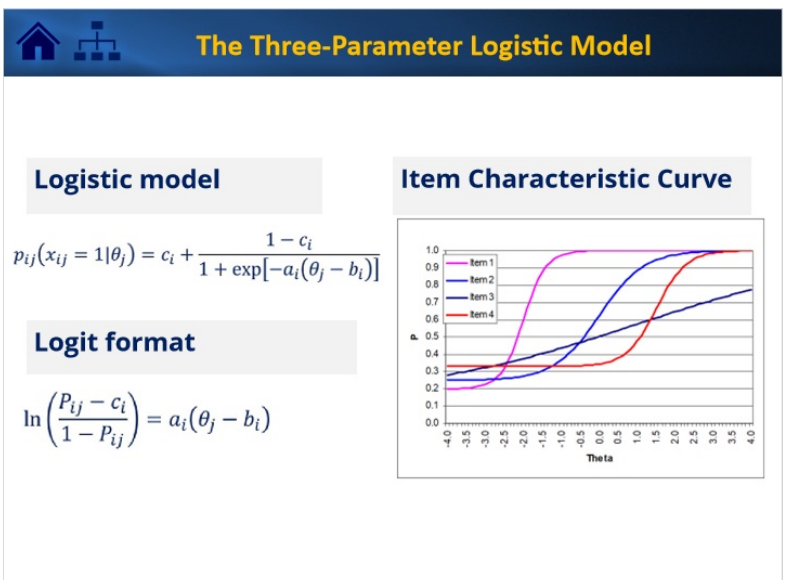
Item Characteristic Curve



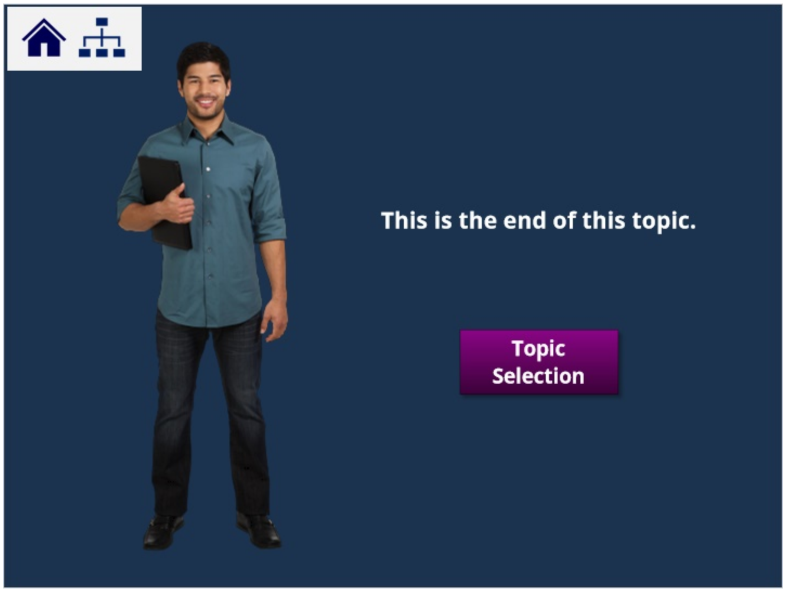
2.10 The Two-Parameter Logistic Model



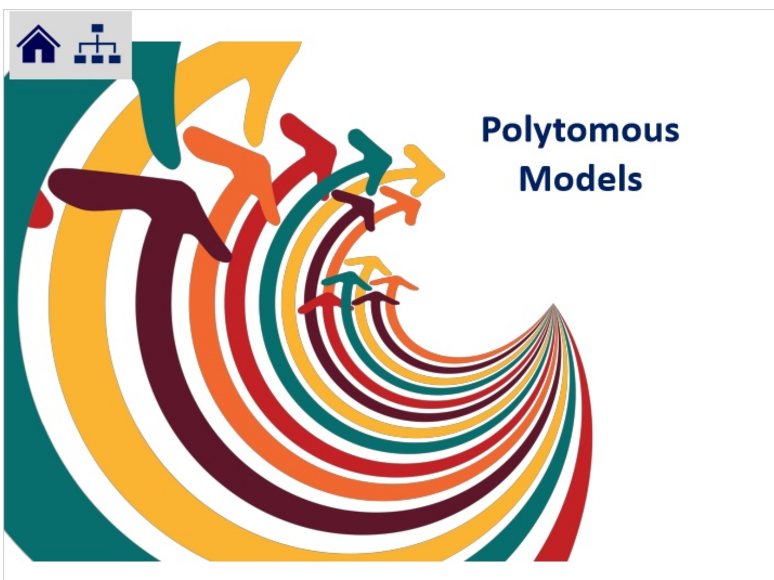
2.11 The Three-Parameter Logistic Model



2.12 Bookend: Dichotomous Models



2.13 Bookmark: Polytomous Models



2.14 Overview of Different Polytomous Models

Overview of Polytomous Models

Adjacent-category models

Graded response model	(e.g., Samejima, 1969)
Modified graded response model	(e.g., Muraki, 1990)

Divide-by-total models

Partial credit model	(e.g., Masters, 1982)
Generalized partial credit model	(e.g., Muraki, 1992)
Rating scale model	(e.g., Andrich, 1978)
Modified rating scale model	(e.g., du Toit et al., 1999)
Nominal response model	(e.g., Bock, 1972)

2.15 The Graded Response Model I

The Graded Response Model

2PL IRT model

$$p_{ij}(x_{ij} = 1|\theta_j) = \frac{\exp[a_i(\theta_j - b_i)]}{1 + \exp[a_i(\theta_j - b_i)]}$$

Graded Response Model

- Probability of a person's raw item score x falling in or above a given category *k* = 1, 2, ..., *m* conditional on theta
- Operating characteristic curve

$$p^*_{ix}(\theta) = \frac{\exp[a_i(\theta_j - b_{ik})]}{1 + \exp[a_i(\theta_j - b_{ik})]}$$

- Category threshold represents the trait level to respond at or above threshold k with 0.5 probability.

2.16 The Graded Response Model II

The Graded Response Model

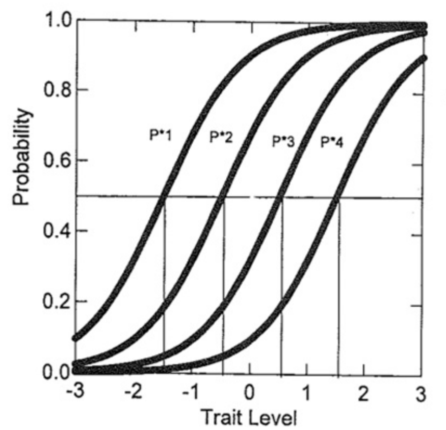


FIG. 5.1. Operating characteristic curves for five-category example item under the graded response model.

2.17 The Graded Response Model III

The Graded Response Model

Probability of a person's raw item score-x fall in or above a given category k=1, 2,..m conditional on theta

$$P_{ix}^{*}(\theta) = \frac{\exp\left[a_i(\theta_j - b_{ik})\right]}{1 + \exp\left[a_i(\theta_j - b_{ik})\right]}$$

Operating characteristic curve

Given three scores x=0, 1, 2 with 3 response options, the item is treated as a series of 3-1=2 dichotomies: 0 vs 1, 2; 0, 1 vs 2

$$P_{ix}(\theta) = P_{ix}^{*}(\theta) - P_{i(x+1)}^{*}(\theta)$$
$$P_{i0}(\theta) = 1 - P_{i1}^{*}(\theta)$$
$$P_{i1}(\theta) = P_{i1}^{*}(\theta) - P_{i2}^{*}(\theta)$$
$$P_{i2}(\theta) = P_{i2}^{*}(\theta) - 0$$

Category response curves: the probability of an examinee responding in a particular category conditional on trait level.

2.18 The Graded Response Model IV

The Graded Response Model

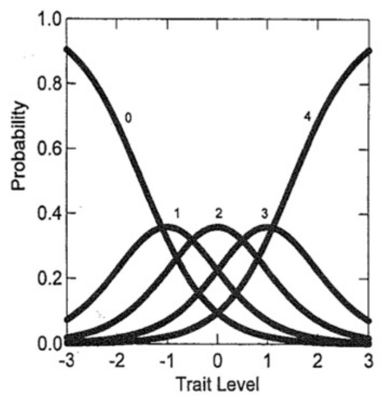


FIG. 5.2. Category response curves for five-category example item under the graded response model.

For any given theta value, the sum of the response probabilities is equal to 1.

2.19 The Partial Credit Model

The Partial Credit Model

$$p_{ix}(\theta) = \frac{\exp\left[\sum_{s=0}^{x}(\theta_j - b_{is})\right]}{\sum_{k=0}^{K}\exp\left[\sum_{s=0}^{k}(\theta_j - b_{is})\right]}$$

where $\sum_{s=0}^{0}(\theta - b_{is}) = 0$

- The point on the latent trait scale at which two consecutive category response curves intersect.
- Relative difficulty of each step.
- Indicates where on the latent trait scale the response of one category becomes relatively more likely than the previous category.

Masters (1982)

2.20 The Reparameterized Partial Credit Model I

The Reparameterized Partial Credit Model

Assume three score categories: 0, 1, 2

The probability of getting a certain score is

$t_1 = \exp(\theta_j - b_i - d_{i1})$ $\quad$ $t_2 = \exp(2\theta_j - 2b_i - d_{i1} - d_{i2})$

$$p(x_{ji} = 0) = \frac{1}{1 + t_1 + t_2}$$

$$p(x_{ji} = 1) = \frac{t_1}{1 + t_1 + t_2}$$

$$p(x_{ji} = 2) = \frac{t_2}{1 + t_1 + t_2}$$

2.21 The Reparameterized Partial Credit Model II

Category Response Curves

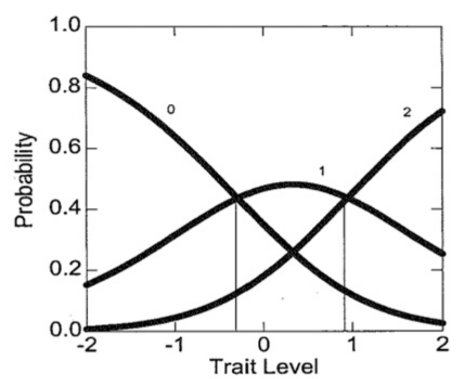
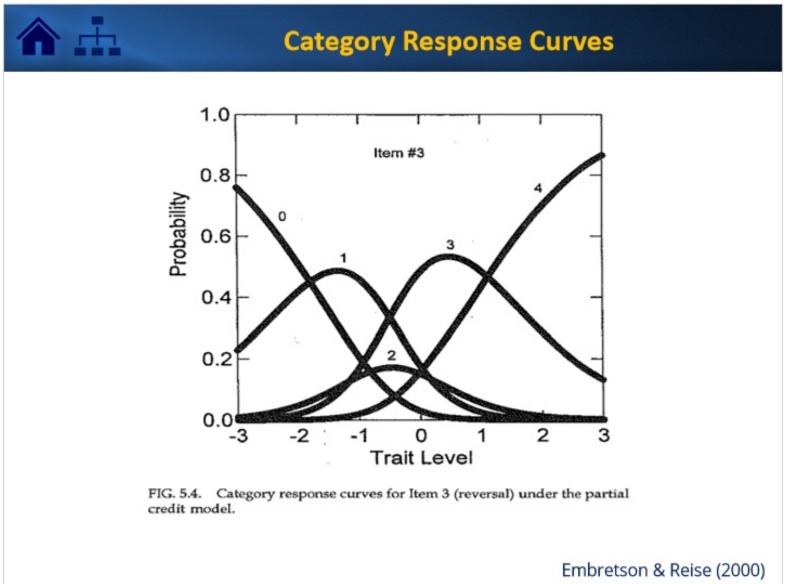


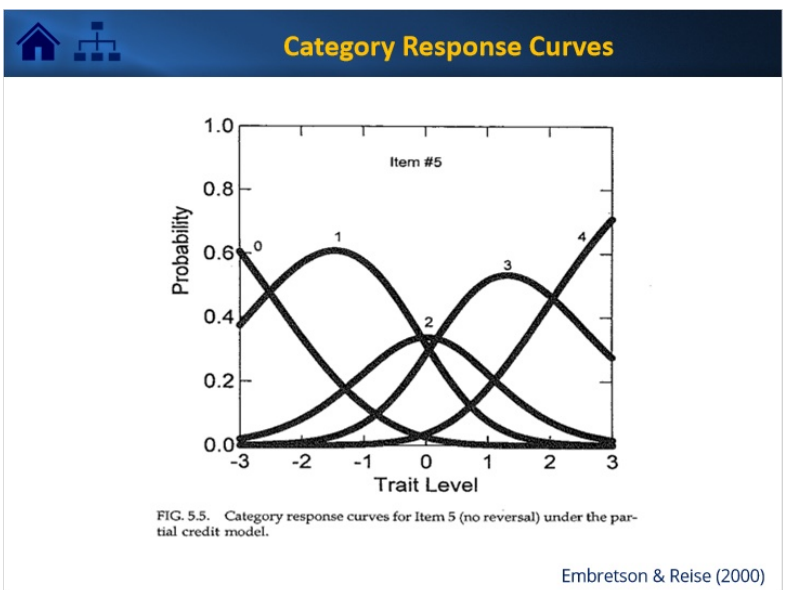
FIG. 5.3. Category response curves for the example items under the partial credit model.

Embretson & Reise (2000)

2.22 The Reparameterized Partial Credit Model III



2.23 The Reparameterized Partial Credit Model IV



2.24 The Rating Scale Model

The Rating Scale Model

$$p_{ix}(\theta) = \frac{\exp\left[\sum_{s=0}^{x}\left(\theta_j - (d_i + c_s)\right)\right]}{\sum_{k=0}^{K}\exp\left[\sum_{s=0}^{k}\left(\theta_j - (d_i + c_s)\right)\right]}$$

Category intersection parameter

Item location parameter

where $\sum_{s=0}^{0}(\theta - (d_i + c_s)) = 0$

Andrich (1978a, 1978b)

2.25 The Generalized Partial Credit Model I

The Generalized Partial Credit Model

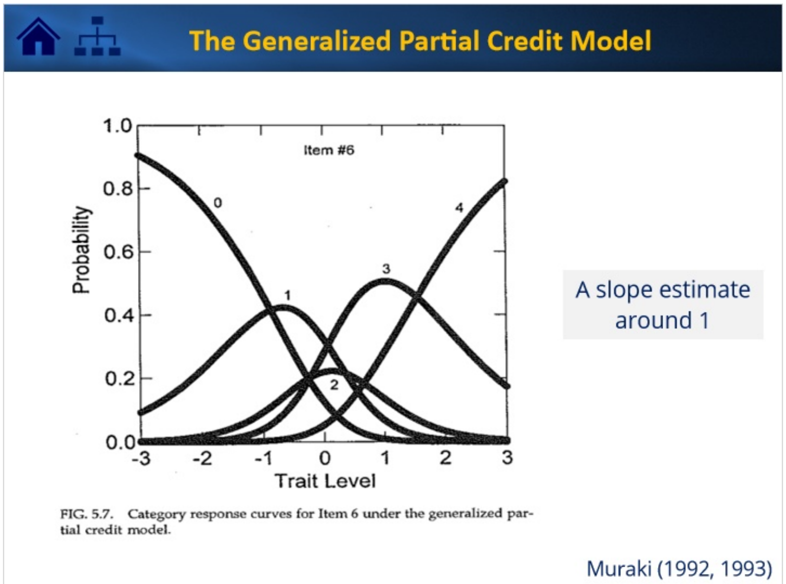
$$p_{ix}(\theta) = \frac{\exp\left[\sum_{s=0}^{x} a_i\left(\theta_j - b_{is}\right)\right]}{\sum_{k=0}^{K}\exp\left[\sum_{s=0}^{k} a_i\left(\theta_j - b_{is}\right)\right]}$$

where $\sum_{s=0}^{0} a_i(\theta - b_{is}) = 0$

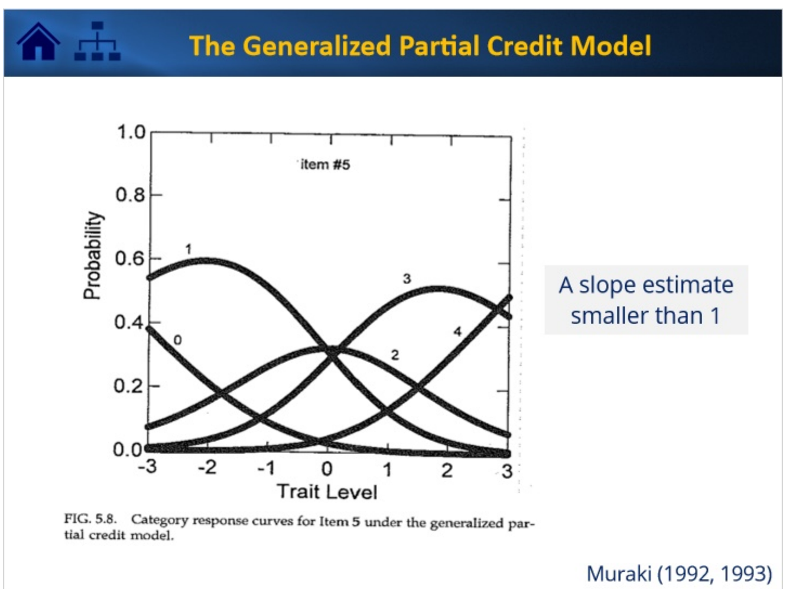
- The point on the latent trait scale at which two consecutive category response curves intersect.
- Indicates where on the latent trait scale the response of one category becomes relatively more likely than the previous category.
- The slope parameter indicate the degree to which categorical responses vary among items as theta level changes.

Muraki (1992, 1993)

2.26 The Generalized Partial Credit Model II



2.27 The Generalized Partial Credit Model III



2.28 The Generalized Partial Credit Model IV

The Generalized Partial Credit Model

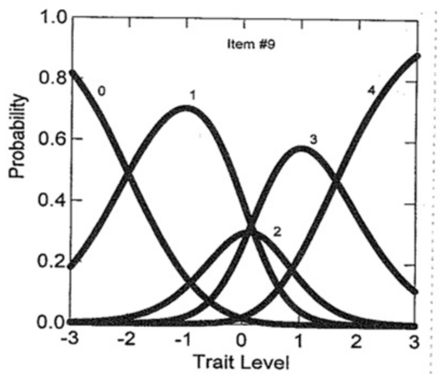


FIG. 5.9. Category response curves for Item 9 under the generalized partial credit model.

A slope estimate larger than 1

Muraki (1992, 1993)

2.29 Logit vs. Probit Models I

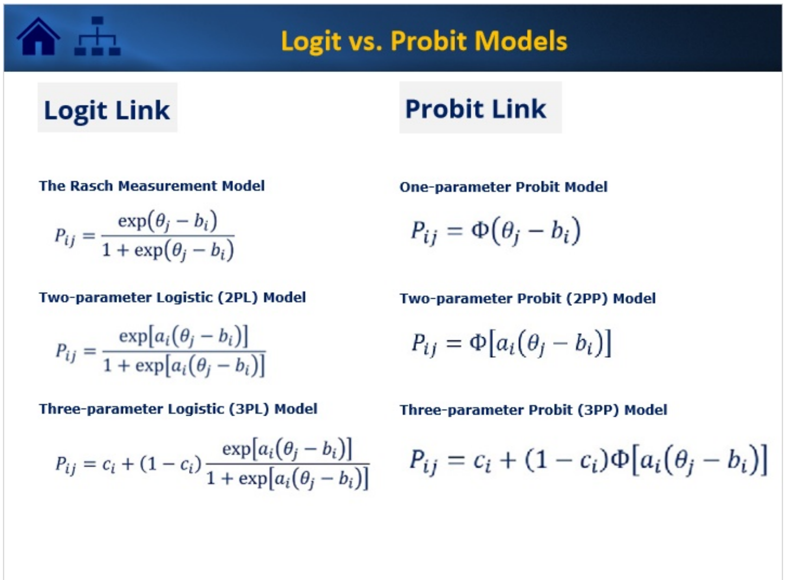
Logit vs. Probit Models

- Both logit and probit link functions can be used in IRT modeling.
- The difference between the probit and logit probabilities are minimal.

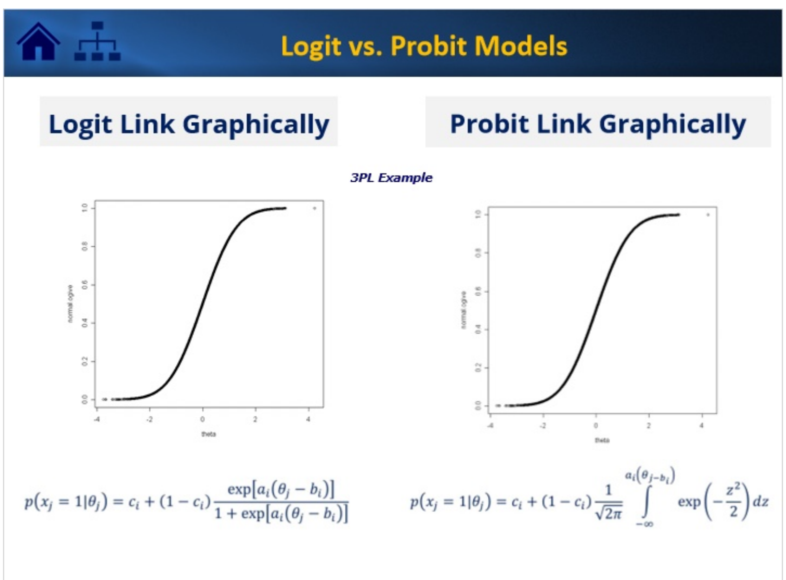
$$|\Phi(x) - \Psi(1.7x)| < .01 \qquad \text{for all } x$$

- By adding a scaling constant of 1.7, the probit link function can approximate the probit link function.

2.30 Logit vs. Probit Models II



2.31 Logit vs. Probit Models III



2.32 Probit Approximation for Logit Link

Probit Approximation for Logit Link

One-parameter Logistic Model (but approximately on the probit scale)

$$P_{ij} = \frac{\exp[1.7(\theta_j - b_i)]}{1 + \exp[1.7(\theta_j - b_i)]}$$

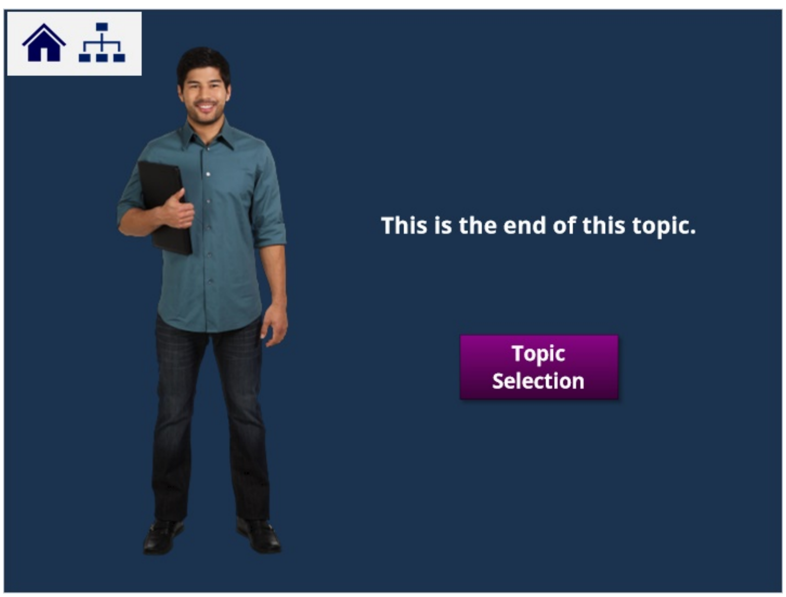
Two-parameter Logistic (2PL) Model (but approximately on the probit scale)

$$P_{ij} = \frac{\exp[1.7a_i(\theta_j - b_i)]}{1 + \exp[1.7a_i(\theta_j - b_i)]}$$

Three-parameter Logistic (3PL) Model (but approximately on the probit scale)

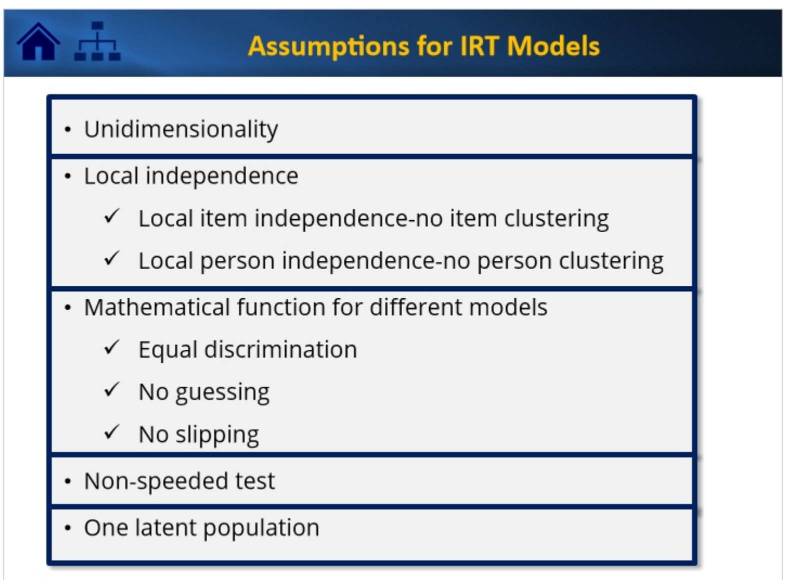
$$P_{ij} = c_i + (1 - c_i)\frac{\exp[1.7a_i(\theta_j - b_i)]}{1 + \exp[1.7a_i(\theta_j - b_i)]}$$

2.33 Bookend: Polytomous Models



2.34 Bookmark: Model Assumptions

2.35 Assumptions for IRT Models



2.36 Local Independence

Local Independence

Local item independence

$$p(U = u_{ij}|\theta_j) = \prod_{i=1}^{I} p(u_{ij}|\theta_j) = p(u_1|\theta_j)p(u_2|\theta_j)\ldots p(u_I|\theta_j)$$

Local person independence

$$p(U_i = u_{ij}|\theta_j) = \prod_{j=1}^{n} p(u_{ij}|\theta_j) = p(u_{i1}|\theta_1)p(u_{i2}|\theta_2)\ldots p(u_{in}|\theta_n)$$

Local independence

$$p(U = u_{ij}|\theta_j) = \prod_{j=1}^{n} \prod_{i=1}^{I} p(u_{ij}|\theta_j)$$

(Embretson & Reise, 2000; Reckase, 2009)

2.37 Violations of Local Independence

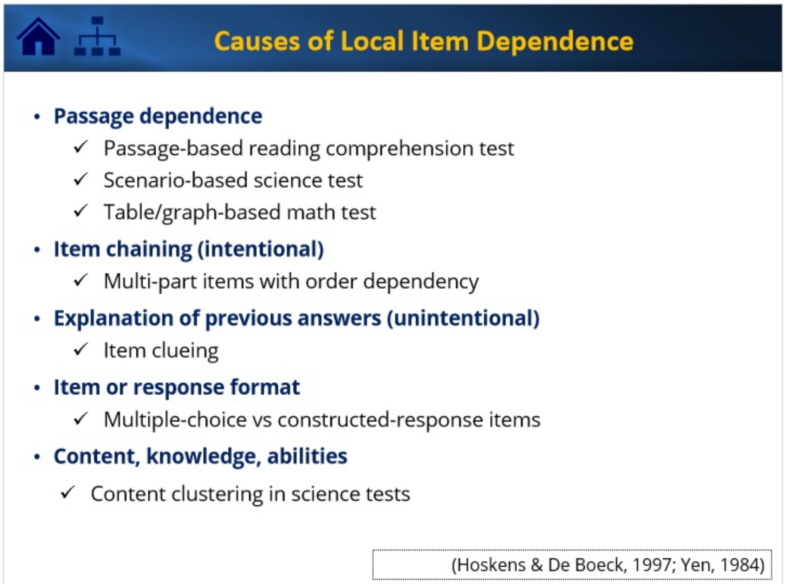
Violations of Local Independence

- **Violations of local item independence**
  - ❑ Item clustering due to manifest item grouping variables
    - ✓ Passage dependence
    - ✓ Item chaining
  - ❑ Latent item grouping variables
    - ✓ Latent item grouping measuring different dimensions of a latent construct

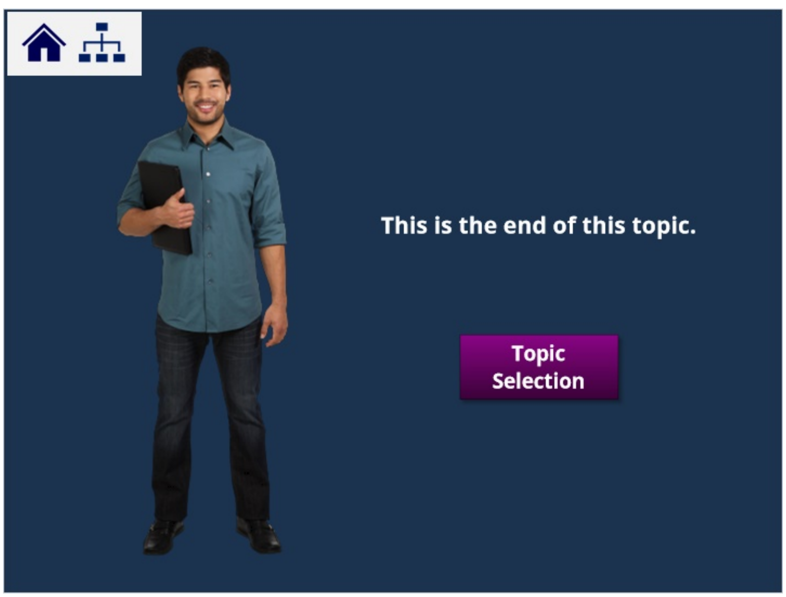
- **Violations of local person independence**
  - ❑ Person clustering due to observed grouping variables
    - ✓ Cluster sampling
    - ✓ Stratified sampling
  - ❑ Person clustering due to unknown latent grouping variables
    - ✓ Different problem-solving strategies
    - ✓ Latent differential item functioning

This ITEMS focuses on local item dependence!

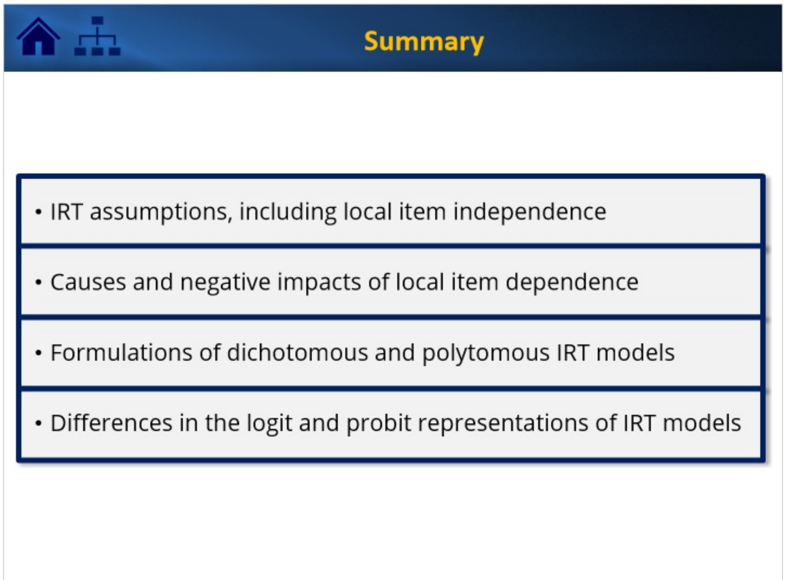
2.38 Causes of Local Item Dependence



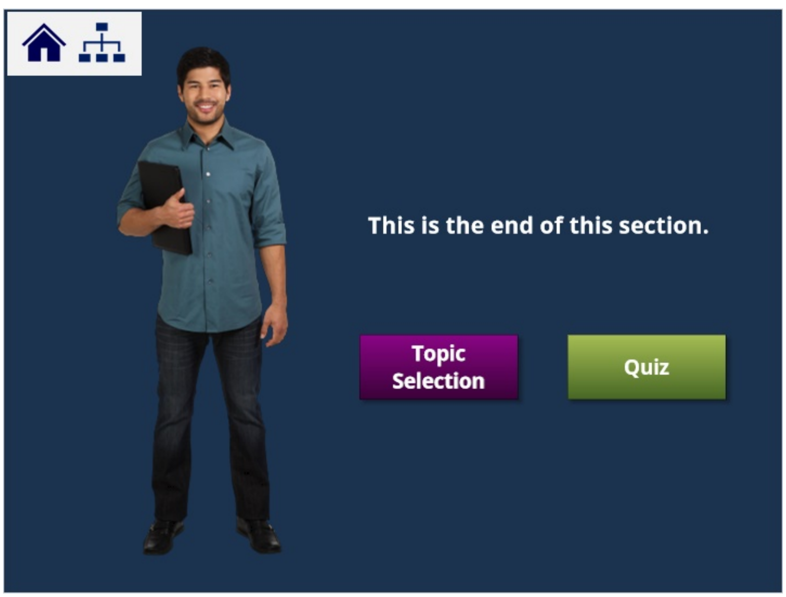
2.39 Bookend: Model Assumptions



2.40 Summary



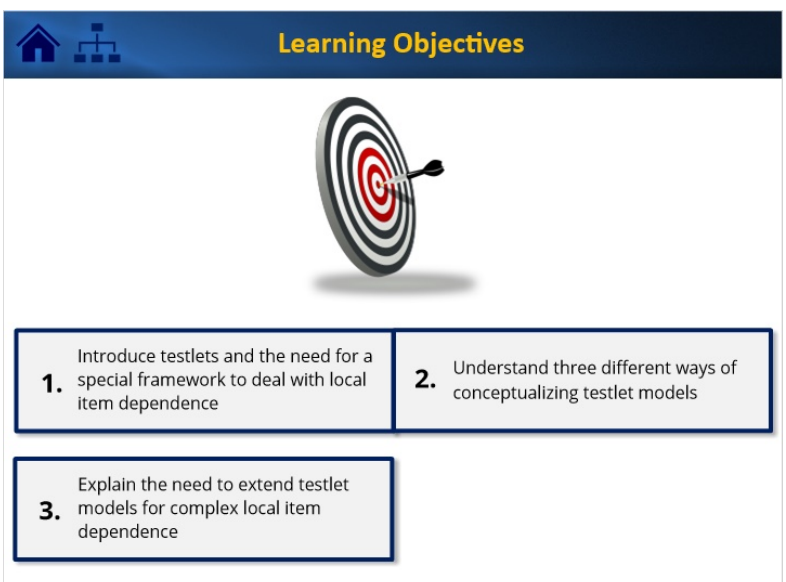
2.41 Bookend: Section 1



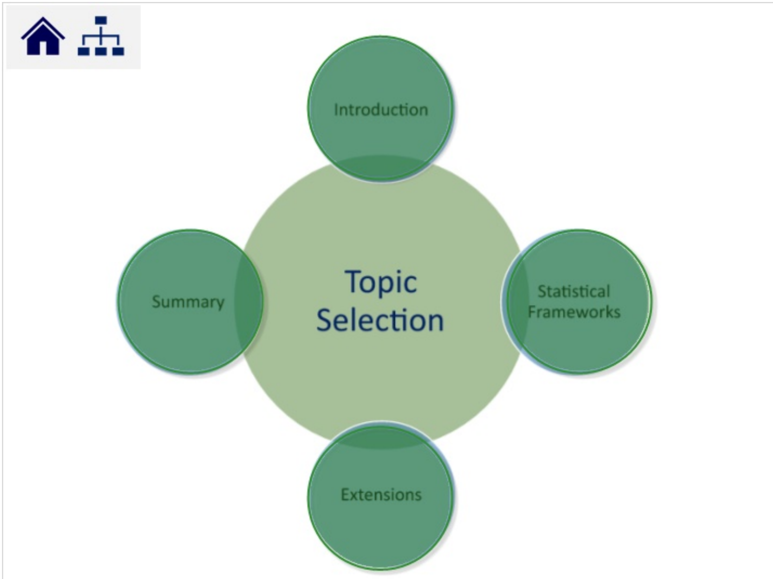
3. Section 2: Testlet Models

3.1 Cover: Section 3

3.2 Learning Objectives: Section 3



3.3 Topic Selection



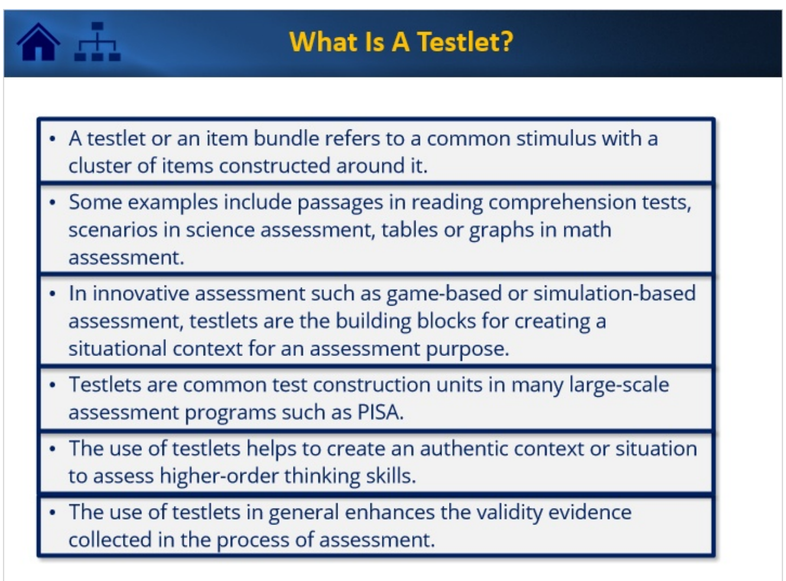
References (Slide Layer)

References

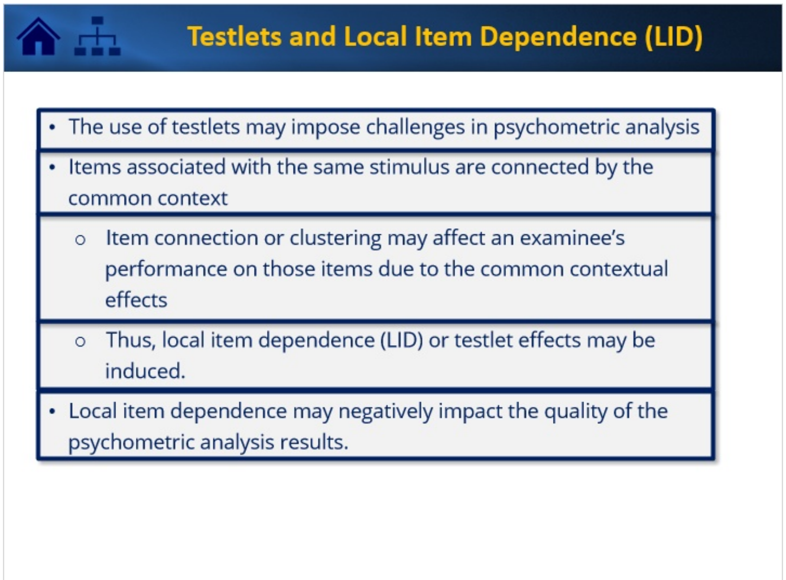
- Hoskens and De Boeck (1997) and Tuerlinckx and De Boeck (1999) modeled item main effects and item interaction effects to account for LID: a constant interaction model and a dimension-dependent interaction model.
- Ip (2002) set up the reproducible and nonreproducible local dependence kernels to model LID based on conditional distributions describing multiple item responses as a function of ability without assuming local independence.
- Wang, Cheng, and Wilson (2005) used a multidimensional item response model to detect specific forms of LID for items across tests connected by common stimuli.
- Bayesian random-effects testlet models (Bradlow, Wainer, & Wang, 1999; Du, 1998; Wainer & Wang, 2000; Wang, Bradlow, & Wainer, 2002) developed to incorporate a parameter into unidimensional item response models, which indicates the interaction between person and item cluster.
- Rasch testlet model (Wang & Wilson, 2005) as a special case of multidimensional random coefficients multinomial logit model.
- Multilevel one-parameter testlet model (Jiao, Wang, & Kamata, 2005) in the framework of hierarchical generalized linear model.

3.4 Bookmark: Introduction

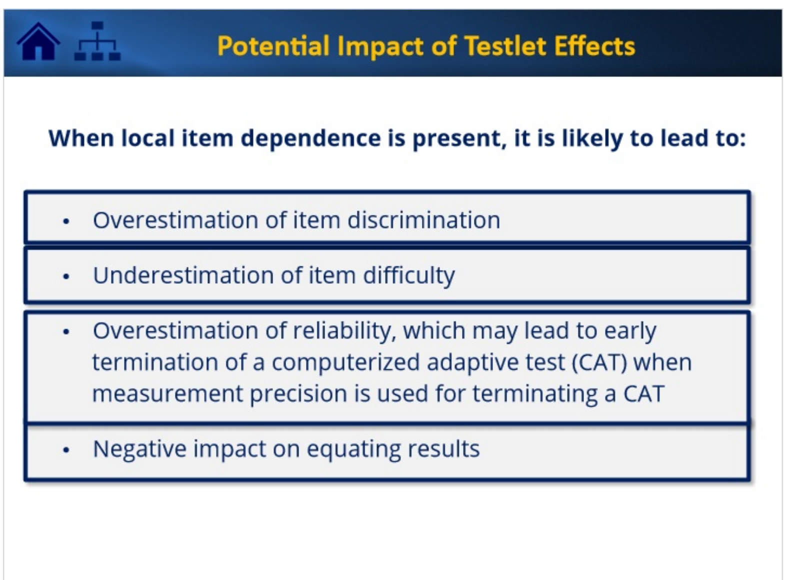
3.5 What Is A Testlet



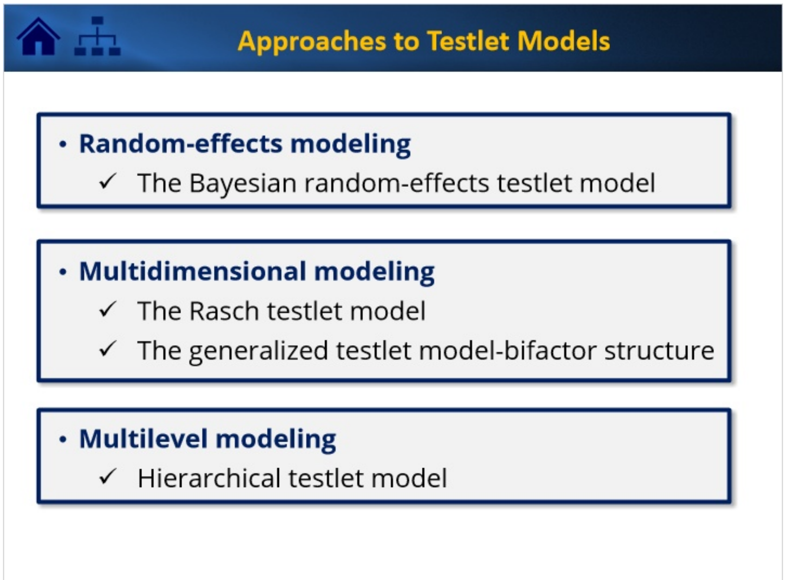
3.6 Testlets and LID



3.7 Potential Impact of Testlet Effects



3.8 Approaches to Testlet Models



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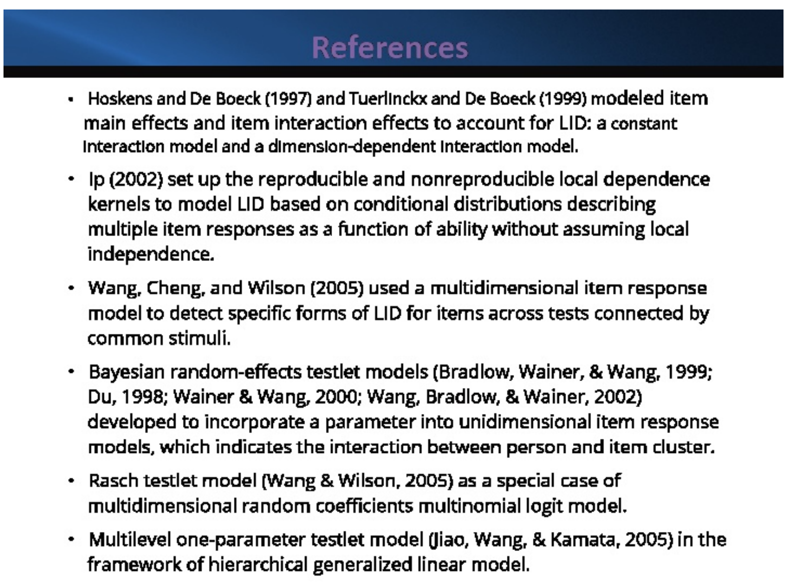
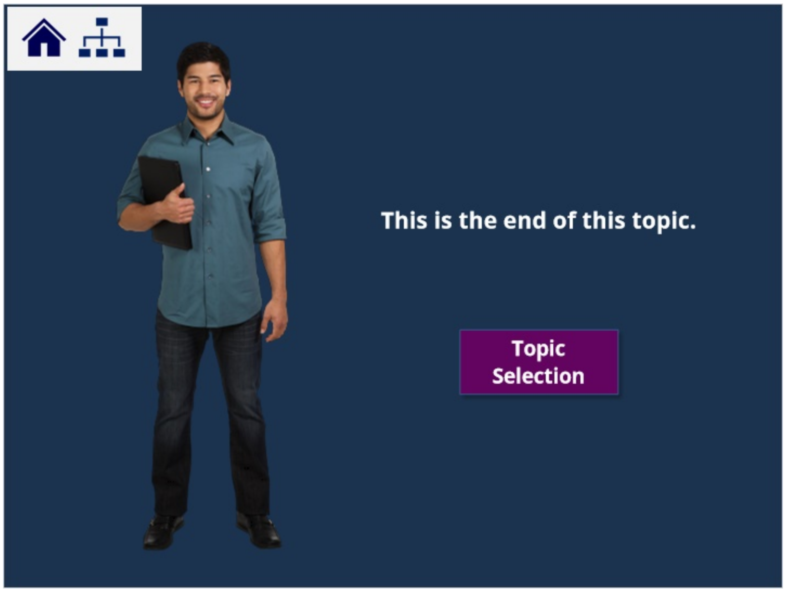


3.9 Bookend: Introduction



3.10 Bookmark: Statistical Frameworks

3.11 The Random-Effects Testlet Model

The Random-Effects Testlet Model

- The two-parameter random-effects testlet model extends the 2PL IRT model by adding a random-effects parameter using probit link function (Bradlow, Wainer, & Wang, 1999)

$$P_{ij} = \Phi\left[a_i\left(\theta_j - b_i - \gamma_{jd(i)}\right)\right]$$

- With logit link function:

$$p_{ij}\left(x_{ij} = 1 \mid \theta_j, \gamma_{jd(i)}, a_i, b_i\right) = \frac{1}{1 + \exp\left[-a_i\left(\theta_j - b_i - \gamma_{jd(i)}\right)\right]}$$

- The testlet effect parameter, $\gamma_{jd(i)}$, is an interaction between person and a testlet
  - Constant for items associated with the same testlet
  - Magnitude of testlet effect quantified by its variance

3.12 The Two-Parameter Logistic Testlet Model

2PL IRT vs 2PL Testlet Model

2PL IRT

$$P(Y_{ij} = 1) = \text{logit}^{-1}(t_{ij})$$
$$t_{ij} = a_i(\theta_j - b_i)$$

2PL Testlet Model

$$P(Y_{ij} = 1) = \text{logit}^{-1}(t_{ij})$$
$$t_{ij} = a_i(\theta_j - b_i - \boxed{\gamma_{jd(i)}})$$

3.13 The Rasch Testlet Model I

The Rasch Testlet Model

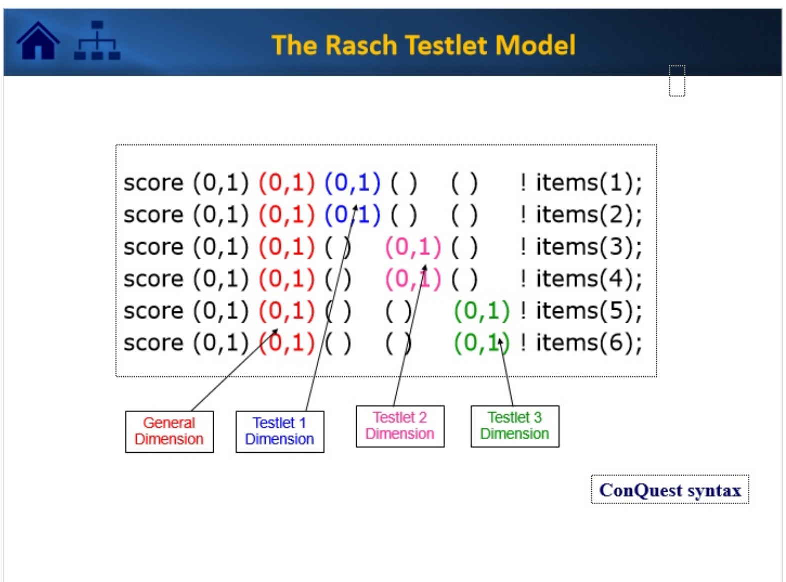
- Formulated from a multidimensional perspective
- Proposed as a special case of multidimensional random coefficients multinomial logit model by adding an additional dimension related to each testlet

(Wang & Wilson, 2005)

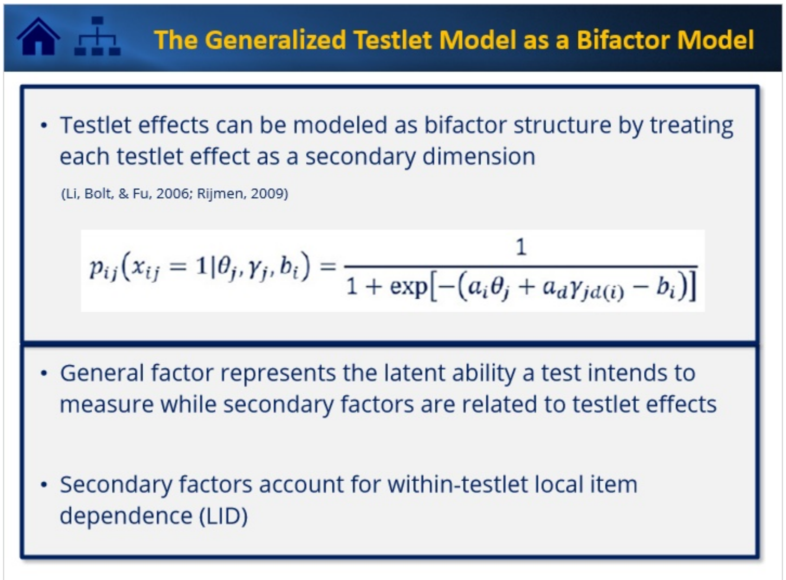
$$p_{ij}(x_{ij} = 1|\theta_j, \gamma_j, b_i) = \frac{1}{1 + \exp[-(\theta_j + \gamma_{jd(i)} - b_i)]}$$

- The ConQuest syntax on the next slide graphically shows the formulation of the multidimensional structure

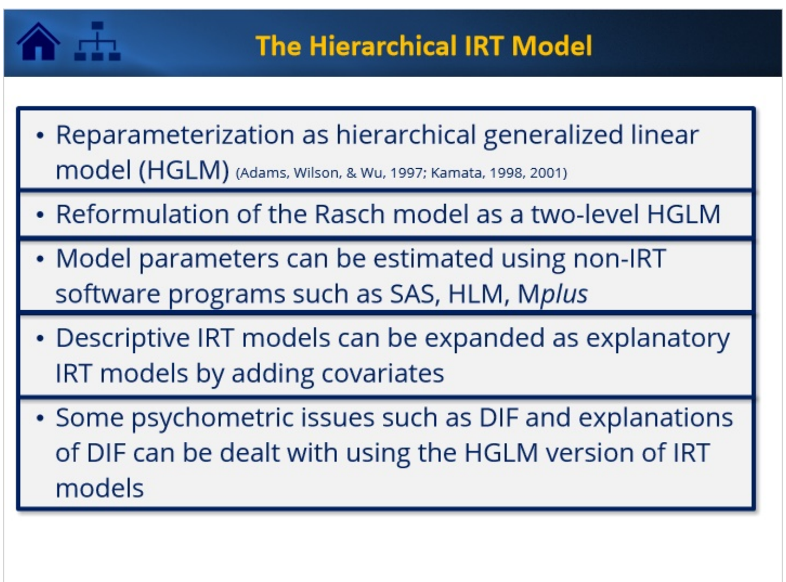
3.14 The Rasch Testlet Model II



3.15 The Generalized Testlet Model as a Bifactor Model

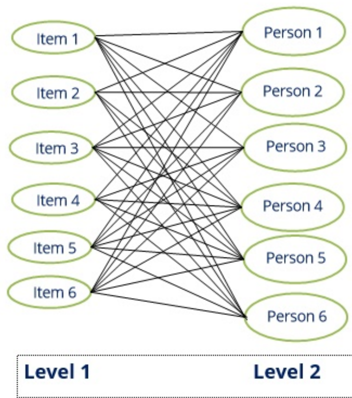


3.16 The Hierarchical IRT Model



3.17 A Two-Level Rasch Equivalent Model

A Two-Level Rasch Equivalent Model



3.18 Parameterization of the Two-Level IRT Model

Parameterization of the 2-Level IRT Model

Level 1 model-item level

$$\log\left(\frac{p_{ij}}{1-p_{ij}}\right) = \beta_{0j} + \sum_{q=1}^{k-1} \beta_{qj} x_{qij}$$

Level 2 model-person level

$$\beta_{0j} = \gamma_{00} + \mu_{0j} \qquad \beta_{qj} = \gamma_{q0}$$

Design matrix

1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

The combined model

$$P_{ij} = \frac{1}{1 + \exp\left[-\left(\mu_{0j} - \left(-\gamma_{00} - \gamma_{q0}\right)\right)\right]}$$

The Rasch model

$$p_{ij}\left(x_{ij} = 1 | \theta_j, b_i\right) = \frac{1}{1 + \exp[-(\theta_j - b_i)]}$$

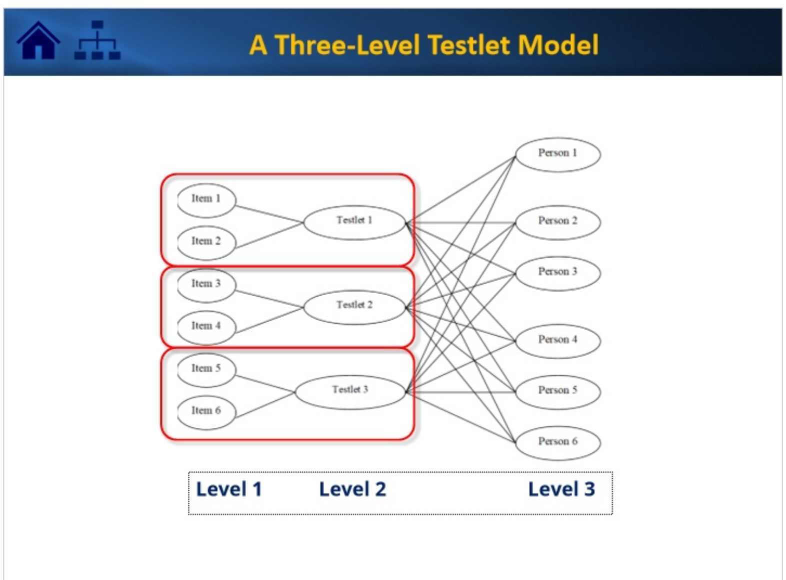
3.19 The Hierarchical Testlet Model

The Hierarchical Testlet Model

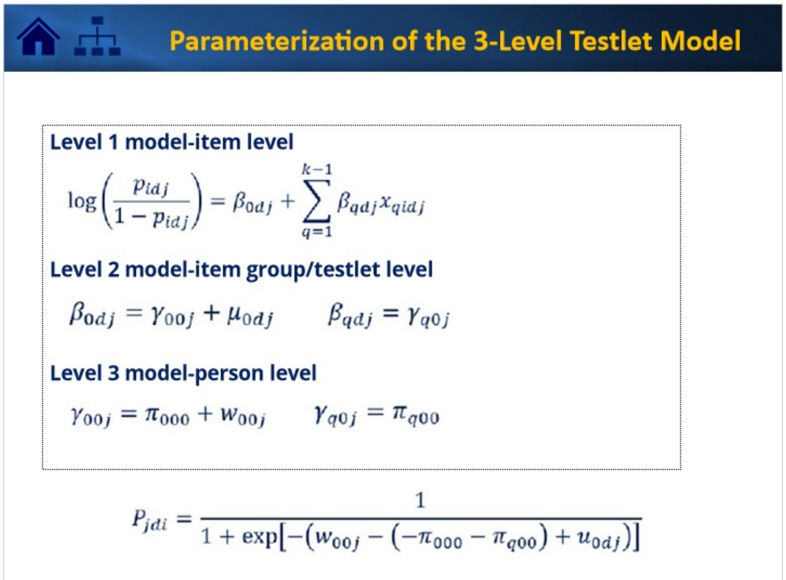
- Model testlet effects as item clustering in a hierarchical generalized linear model (Jiao, Wang, & Kamata, 2005)

- The reformulation of the Rasch testlet model as a three-level HGLM

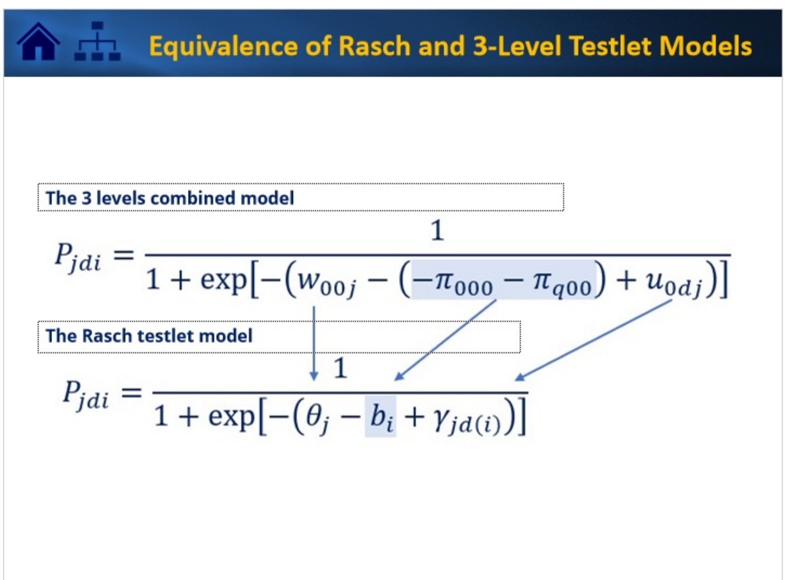
3.20 A Three-Level Testlet Model



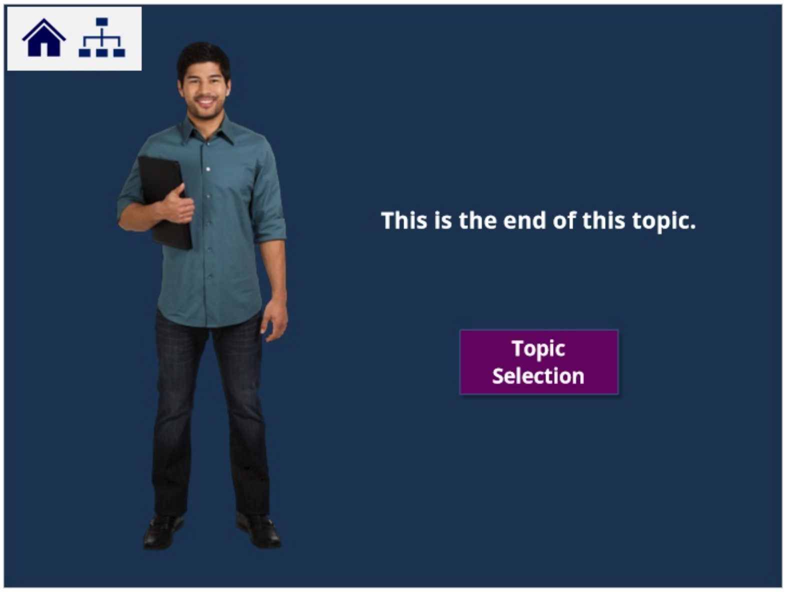
3.21 Parameterization of the Three-Level Testlet Model



3.22 Equivalence of Rasch and Three-Level Testlet Models

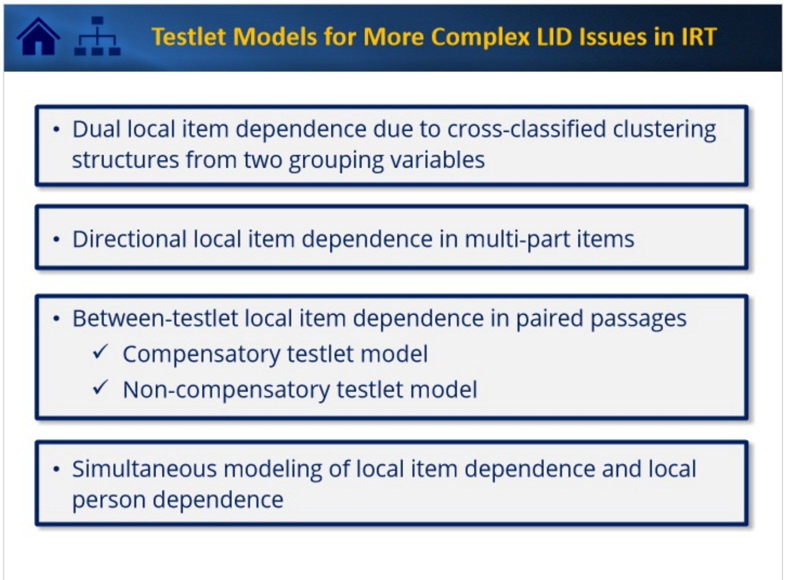


3.23 Bookend: Statistical Frameworks

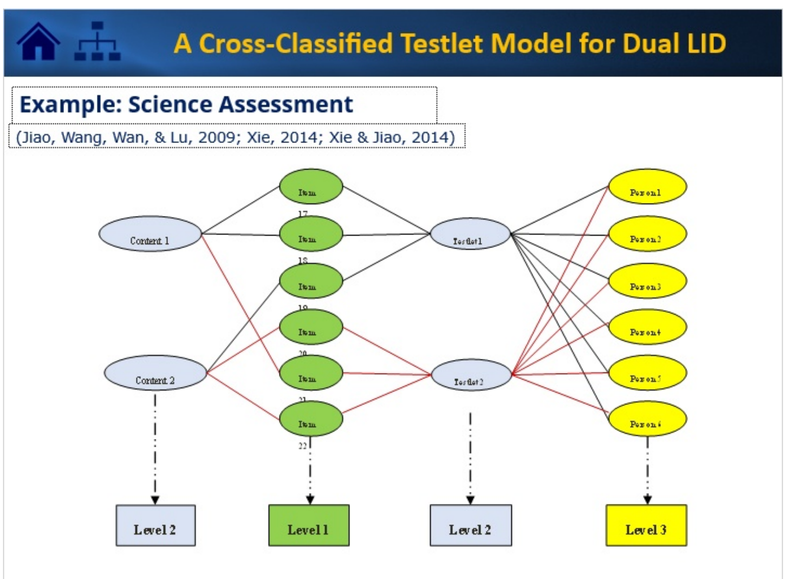


3.24 Bookmark: Extensions

3.25 Testlet Models for More Complex Local Item Dependence Issues in Item Response Modeling



3.26 A Cross-Classified Testlet Model for Dual LID I



3.27 A Cross-Classified Testlet Model for Dual LID II

A Cross-Classified Testlet Model for Dual LID

- Items in a scenario-based science assessment might cluster due to the use of scenarios and content clustering such as physics, life science, earth science

- Items are cross-classified by scenarios and content domains

- Two testlet effect parameters, $\gamma_{js(i)}$ and $\gamma_{jc(i)}$, are added into the model to account for each type of testlet effects due to scenario and content domain

$$p_{jdi} = \frac{1}{1 + \exp[-(\theta_j - b_i + \gamma_{js(i)} + \gamma_{jc(i)})]}$$

3.28 Testlet Models for More Complex LID Issues in IRT

Testlet Models for More Complex LID Issues in IRT

- Directional local item dependence in multi-part items
  - ✓ New item type developed for PARCC consortium tests, also called evidence-based selected response (EBSR)
  - ✓ Starts with a traditional selected-response item followed by a second selected-response item that asks students to show evidence from the text to support their answer to the first item

- A conditional IRT model for directional local item dependence based on item splitting (Marais & Andrich, 2008; Liao, Jiao, & Lissitz, 2016)
  - ✓ Proposed for context-dependent items (Baltrunas & Ricci 2009a & 2009b)
  - ✓ See also for response dependence (Marais & Andrich, 2008; Andrich & Kreiner, 2010; Andrich, Humphry, & Marais, 2012)

3.29 Item Splitting I

Item Splitting

Student	Item 1	Item 2 – Original	Item 2 – Recoded	
			Item 2a	Item 2b
1	1	1	1	x
2	0	1	x	1
3	0	0	x	0
4	0	1	x	1
5	1	0	0	x
6	1	1	1	x
7	1	1	1	x
8	0	0	x	0
9	1	0	0	x
10	0	1	x	1

3.30 Item Splitting II

Item Splitting

- IRT model for directional LID based on item splitting
  (Marais & Andrich, 2008; Liao, Jiao, & Lissitz, 2016)

$$P(x_{2j} = 1 | x_{1j} = x) = \frac{\exp[a_2(\theta_j - (b_2 - xd))]}{1 + \exp[a_2(\theta_j - (b_2 - xd))]}$$

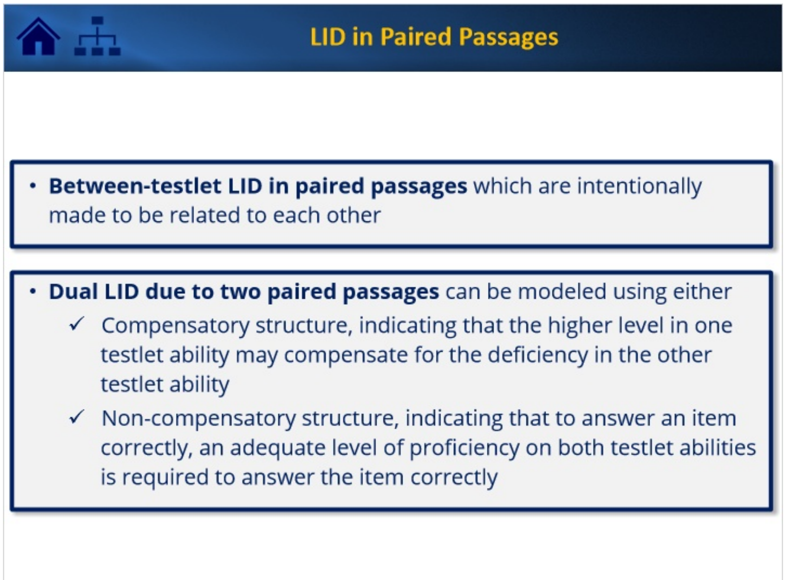
- Probability of second item conditional on getting first item wrong

$$P(x_{2j} = 1 | x_{1j} = 0) = \frac{\exp[a_2(\theta_j - b_2)]}{1 + \exp[a_2(\theta_j - b_2)]}$$

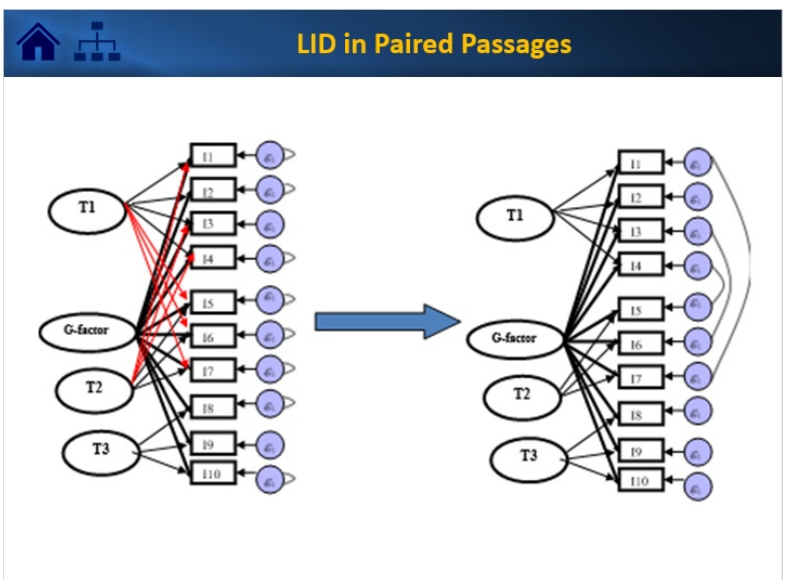
- Probability of second item conditional on getting first item right
  (Includes non-negative parameter *d*)

$$P(x_{2j} = 1 | x_{1j} = 1) = \frac{\exp[a_2(\theta_j - (b_2 - d))]}{1 + \exp[a_2(\theta_j - (b_2 - d))]}$$

3.31 LID in Paired Passages I



3.32 LID in Paired Passages II



3.33 Compensatory and Non-Compensatory Models

Compensatory and Non-Compensatory Models

A Non-Compensatory Two-Parameter Testlet Model

$$P(x_{ij}|a_{1i}, a_{2i}, b_i, \theta_j, \gamma_{jt}) = \prod \frac{1}{1 + \exp[-(a_{1i} * \theta_j + a_{2i} * \gamma_{jt} - b_i)]}$$

A Compensatory Two-Parameter Testlet Model (Jiao & Shu, 2014)

$$P(x_{ij}|a_{1i}, a_{2i}, a_{3i}, b_i, \theta_j, \gamma_{jt1}, \gamma_{jt2}) = \frac{1}{1 + \exp[-(a_{1i} * \theta_j + a_{2i} * \gamma_{jt1} + a_{3i} * \gamma_{jt2} - b_i)]}$$

3.34 Dual LID and LPD

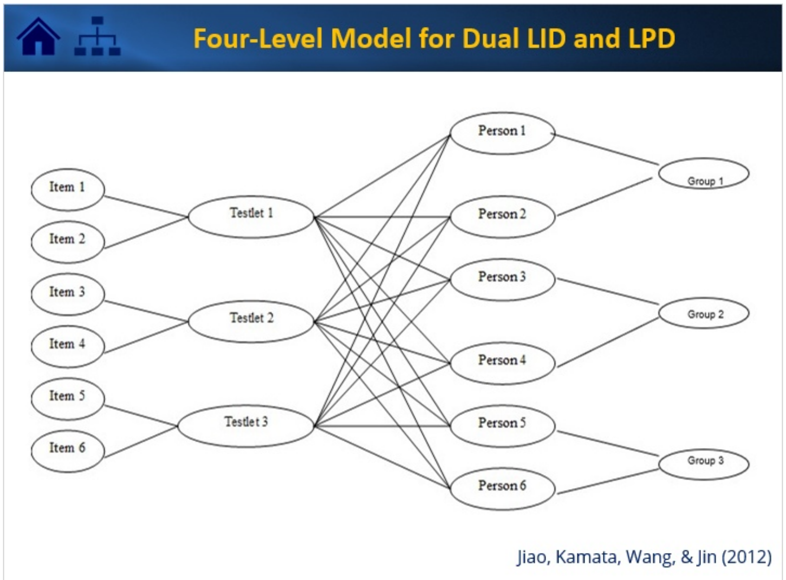
Dual LID and LPD

Simultaneous modeling of local item dependence (LID) and local person dependence (LPD):

- Modeling of local item dependence and local person dependence simultaneously
- Person clustering effect is modeled by adding a group-specific ability parameter, θ_g
- Testlet effects are modeled as in the standard testlet model

$$P(x_{ij}|b_i, \theta_j, \gamma_{jt}, \theta_g) = \frac{1}{1 + \exp[-(\theta_j + \gamma_{jt(i)} + \theta_g - b_i)]}$$

3.35 Four-Level Model for Dual LID and LPD

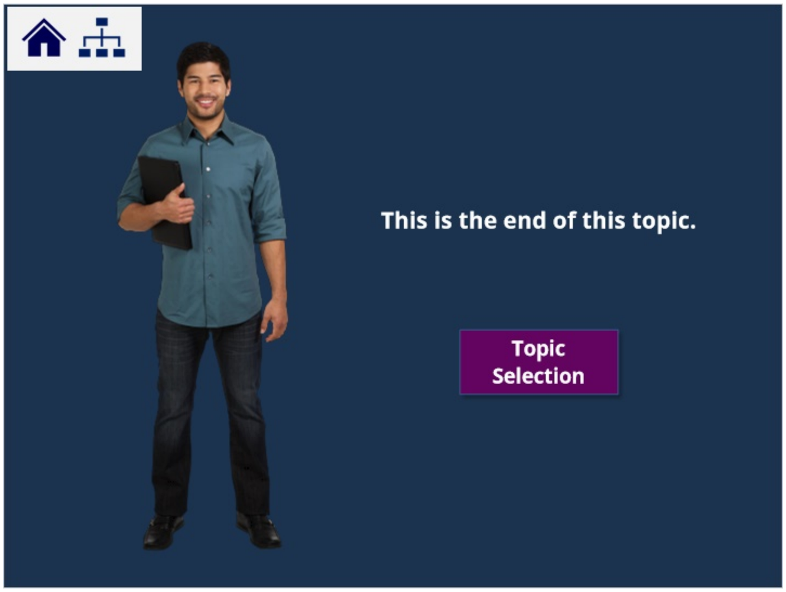


3.36 Cutting-Edge Examples

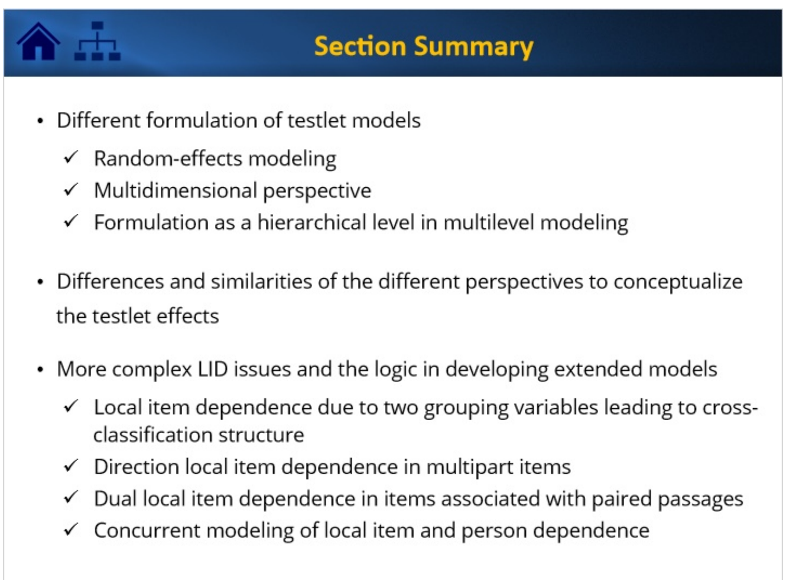
Game- and Simulation-based Assessments

- **Multidimensionality can be modeled using**
  - ✓ Compensatory multidimensional IRT model (Reckase, 2009)
  - ✓ Conjunctive multidimensional IRT model (Embretson, 1997)
  - ✓ Cognitive diagnostic models
- **Local item dependence can be modeled using**
  - ✓ Latent variables (Almond et al., 2009; Levy & Mislevy, 2004)
  - ✓ Bayesian networks for directional LID (Jensen, 1996; Pearl, 1988)
  - ✓ Testlet models for contextual dependence (Bradlow, Wainer, & Wang, 1999)

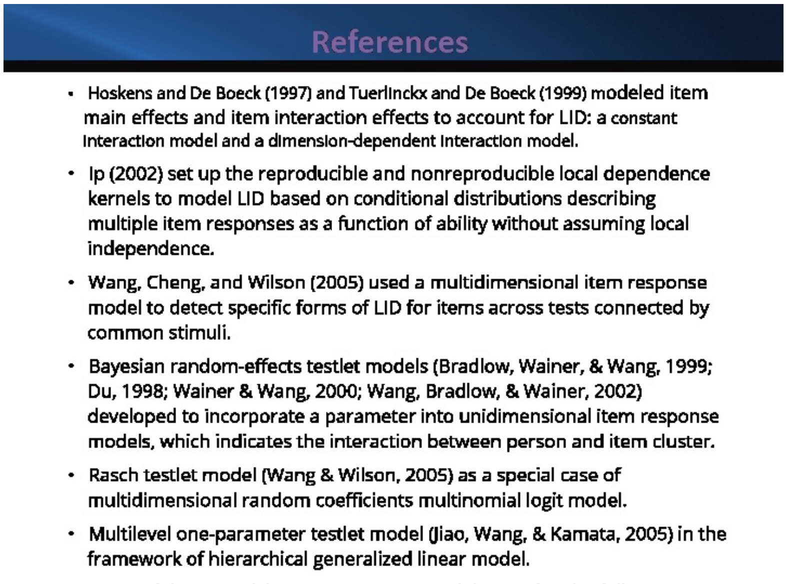
3.37 Bookend: Extensions



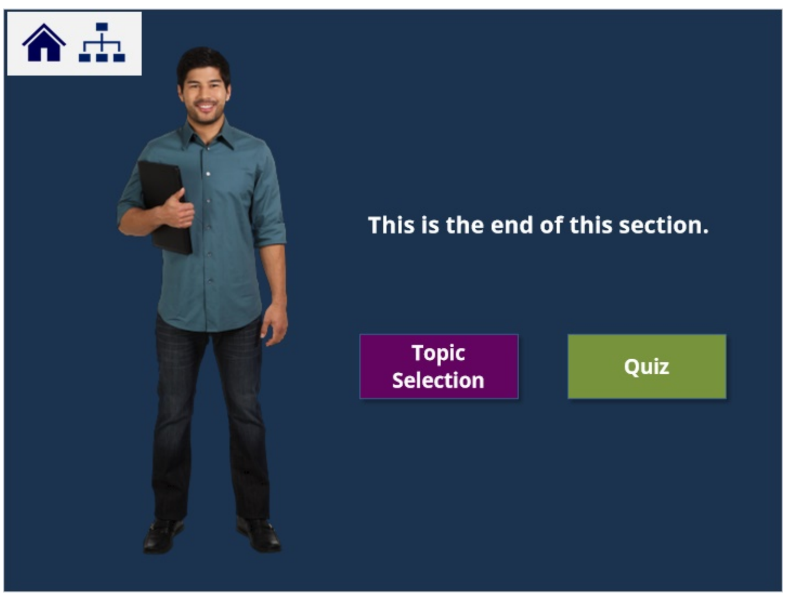
3.38 Section Summary



References (Slide Layer)

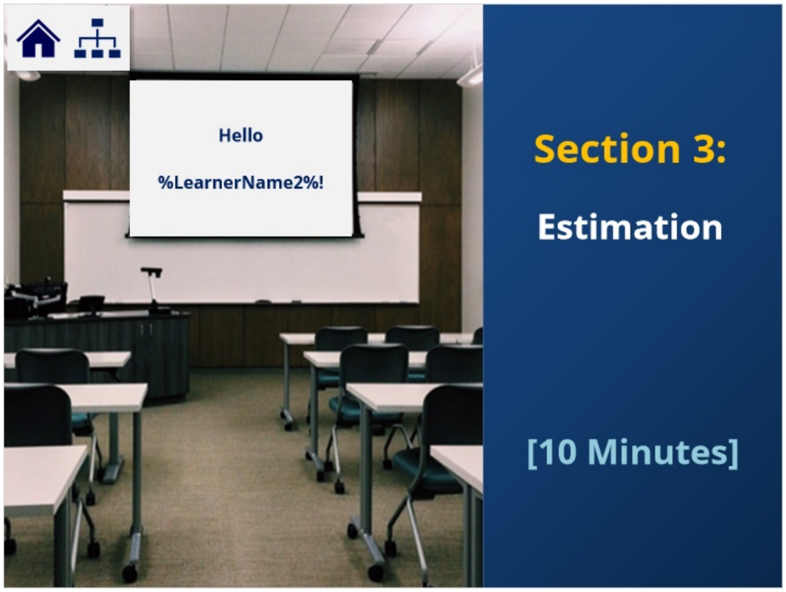


3.39 Bookend: Section 3

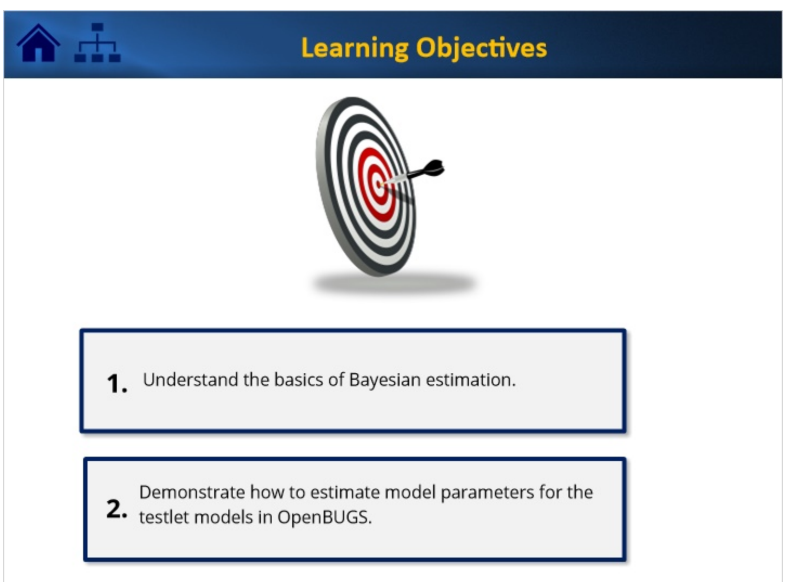


4. Section 3: Estimation

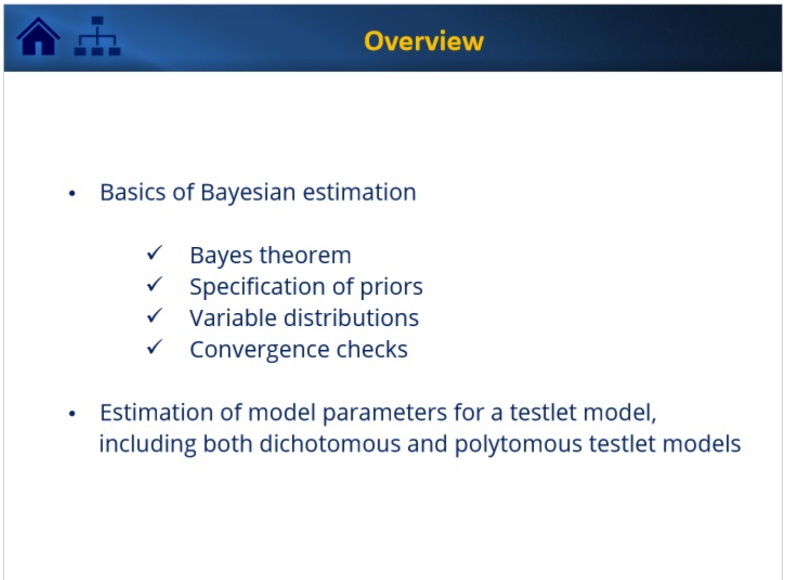
4.1 Cover: Section 3



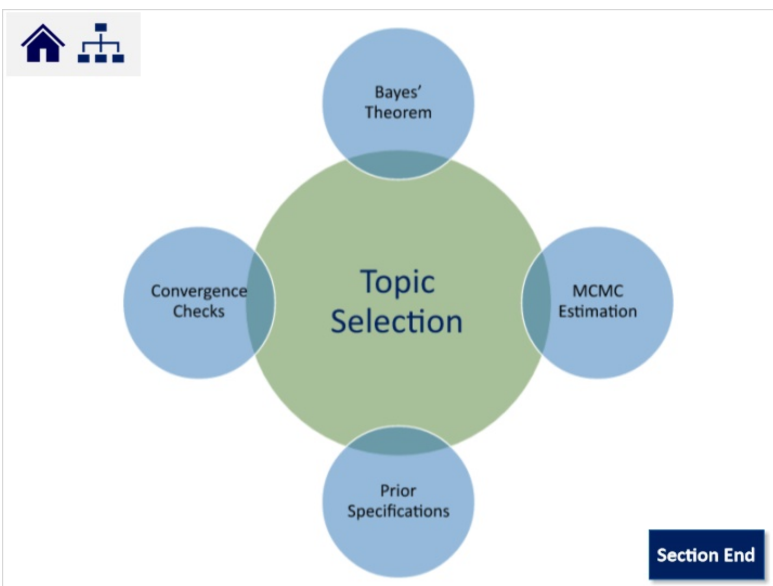
4.2 Learning Objectives: Section 4



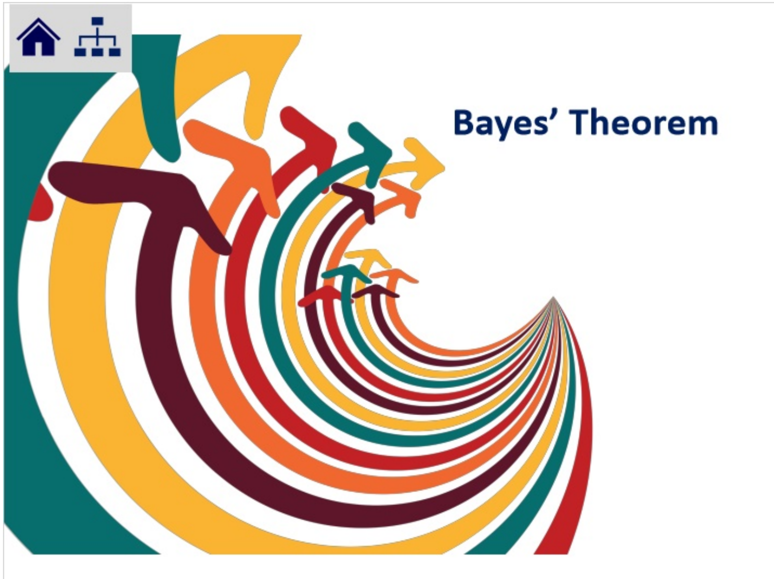
4.3 Summary



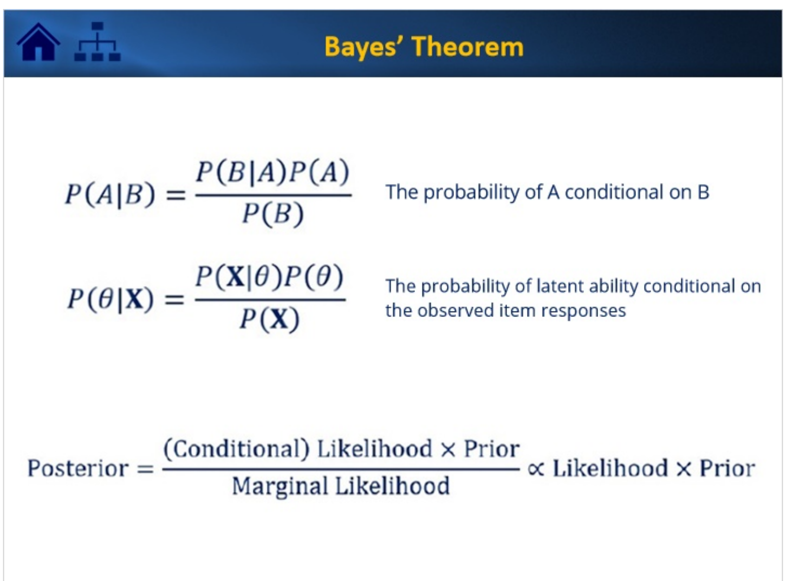
4.4 Topic Selection



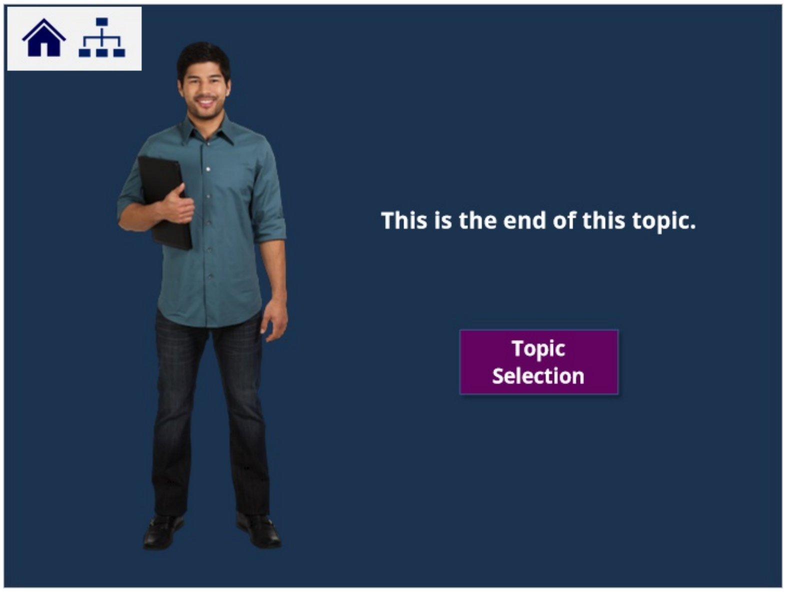
4.5 Bookmark: Bayes' Theorem



4.6 Bayes' Theorem



4.7 Bookend: Bayes' Theorem



4.8 Bookmark: MCMC Estimation

4.9 Markov Chain Monte Carlo (MCMC) Estimation I

Markov Chain Monte Carlo (MCMC) Estimation

- A **Markov chain** is a series of updated posterior distributions based on prior distributions and the likelihood (or data).
- **Model parameter estimates** are repeatedly sampled from their full conditional posterior distributions over a large number of iterations.
- A **sequence of *W*** (a vector of random variables) could be obtained as:

$$\{W^0, W^1, W^2, \cdots, W^t, W^{t+1}\}$$

- At each $t+1$ state, W^{t+1} is sampled from a **conditional distribution** of the current state, $p(W^{t+1}|W^t)$.
- $p(W^{t+1}|W^t)$ is the **transition kernel** $k(W^t|W^{t+1})$.
- The **transition kernel in IRT** is

$$k[(\theta^t, \xi^t), (\theta^{t+1}, \xi^{t+1})] = p(\theta^{t+1}|\xi^t, Y)p(\xi^{t+1}|\theta^{t+1}, Y)$$

4.10 Markov Chain Monte Carlo (MCMC) Estimation II

Markov Chain Monte Carlo (MCMC) Estimation

Similar to **joint maximum likelihood estimation** method where item and person parameters are **estimated iteratively**

There is no need of integration over high-dimensional probability distributions

Time intensive

4.11 Gibbs Sampler

Gibbs Sampler

Step 1: Initiate $\theta_{1,0}, \cdots, \theta_{p,0}$

Step 2: Generate new values at iteration i as follows:

$$\theta_{1,i} \sim \pi(\theta_1 | \theta_{2,i-1}, \theta_{3,i-1}, \cdots, \theta_{p,i-1})$$
$$\theta_{2,i} \sim \pi(\theta_2 | \theta_{1,i}, \theta_{3,i-1}, \cdots, \theta_{p,i-1})$$
$$\vdots$$
$$\theta_{p,i} \sim \pi(\theta_p | \theta_{1,i}, \theta_{2,i}, \cdots, \theta_{p-1,i})$$

Step 3: Repeat until convergence.

θ may denote an unknown parameter of interest, not necessarily the ability parameter in IRT

4.12 Metropolis Hastings Algorithm

Metropolis Hastings Algorithm

Step 1: Establish the set of starting values for the parameters: $\boldsymbol{\theta}^{j=0}$

Step 2: Set $j = j + 1$, and draw a set of candidate values, $\boldsymbol{\theta}^*$, from a proposal distribution

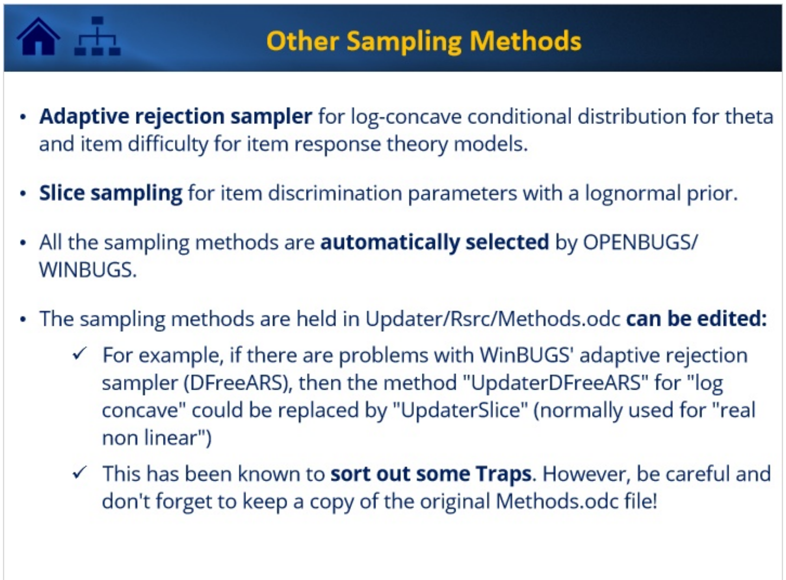
Step 3: Compute the ratio $r = \frac{\pi(\theta^*|y)}{\pi(\theta^{t-1}|y)} \frac{J_t(\theta^{t-1}|\theta^*)}{J_t(\theta^*|\theta^{t-1})}$ and determine whether it is smaller or larger than 1. Choose the smaller of the two and denote it by α

Step 4: Draw a random value u from a uniform distribution defined over (0,1)

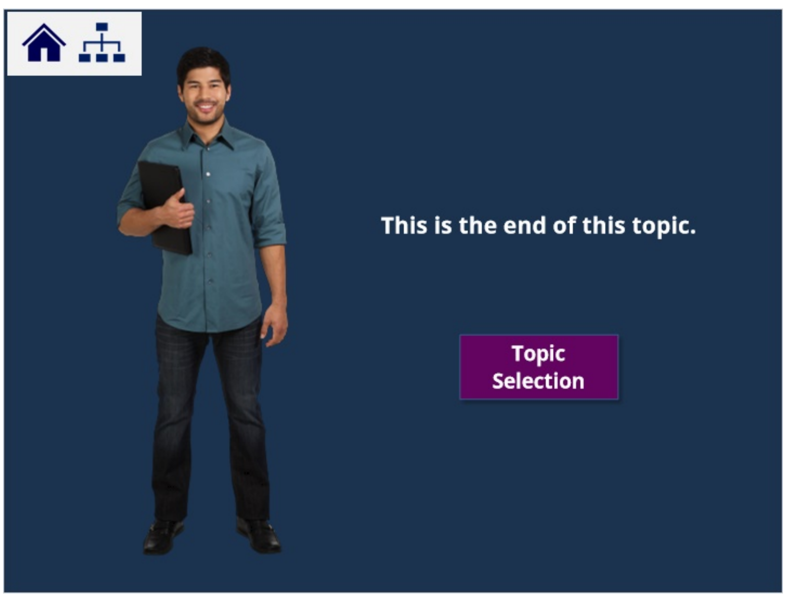
Step 5: Compare α with u. If $\alpha > u$, then set $\boldsymbol{\theta}^j = \theta^*$; otherwise, set $\boldsymbol{\theta}^j = \boldsymbol{\theta}^{j-1}$

Step 6: Return to Step 2 until enough candidate values are sampled

4.13 Other Sampling Methods

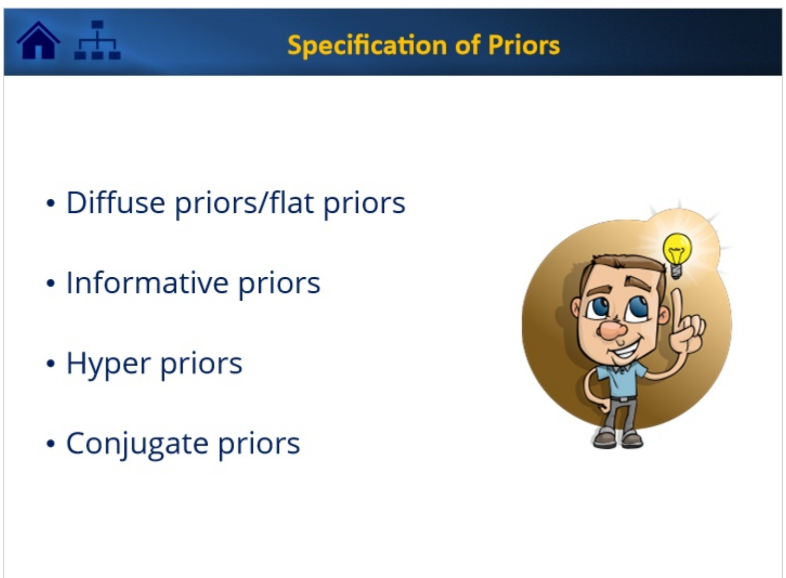


4.14 Bookend: MCMC Estimation

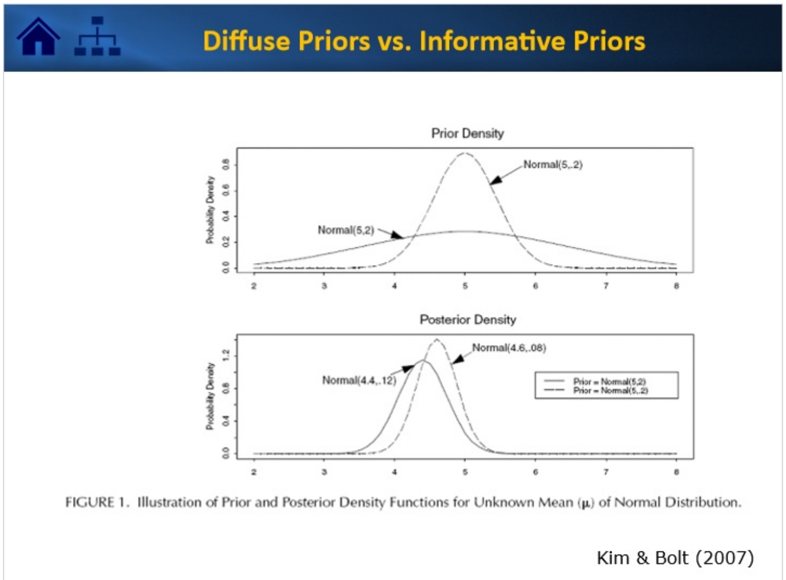


4.15 Bookmark: Prior Specifications

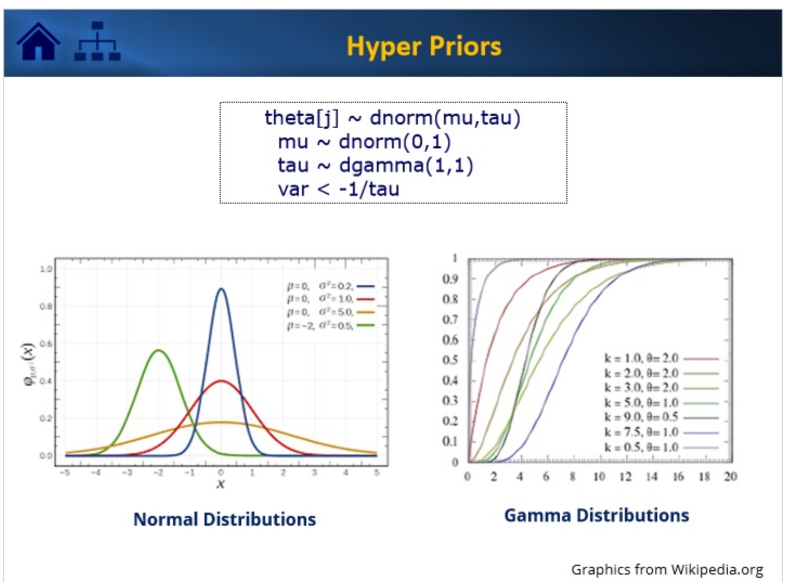
4.16 Specification of Priors I



4.17 Specification of Priors II



4.18 Specification of Priors III



4.19 Specification of Priors IV

Conjugate Priors

TABLE 3
Some Familiar Univariate Conjugate Priors in Exponential Families[a]

Prior $\pi(\theta)$	*Data*	*Posterior* $\pi(\theta \mid y)$
$\pi(\theta) \equiv N(\mu, \tau^2)$	$Y_i \sim N(\theta, \sigma^2)$	$\pi(\theta \mid y) \equiv N(\frac{\tau^2 n\bar{y} + \sigma^2 \mu}{n\tau^2 + \sigma^2}, \frac{\tau^2\sigma^2}{n\tau^2 + \sigma^2})$
$\pi(\theta) \equiv Beta(\alpha, \beta)$	$Y \sim Bin(n, \theta)$	$\pi(\theta \mid y) \equiv Beta(\alpha + y, n + \beta - y)$
$\pi(\theta) \equiv Gamma(\alpha, \beta)$	$Y_i \sim Poisson(\theta)$	$\pi(\theta \mid y) \equiv Gamma(\alpha + n\bar{y}, (n + \beta^{-1})^{-1}$
$\pi(\theta) \equiv InvGamma(\alpha, \beta)$	$Y_i \sim N(\mu, \theta)$	$\pi(\theta \mid y) \equiv InvGamma(\alpha + \frac{n-1}{2}, (\frac{(n-1)s^2}{2} + \beta^{-1})^{-1})$

Note. In general, probability distributions that belong to an exponential family are the only ones with natural conjugate prior distributions and the posterior will involve the sufficient statistic for θ (see Gelman, Carlin, Stern, & Rubin, 1995, p. 38, for details).

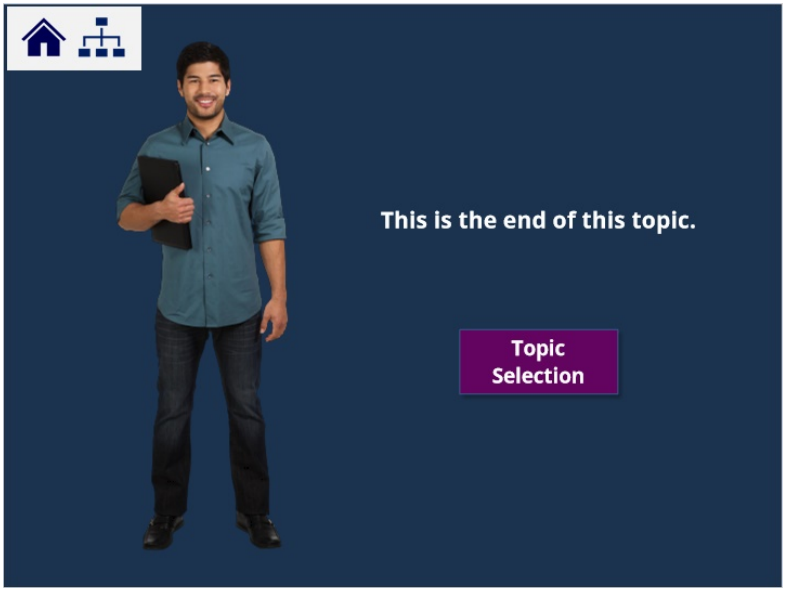
Rupp, Dey, & Zumbo (2004)

4.20 Specification of Priors V

Common Priors for IRT Models

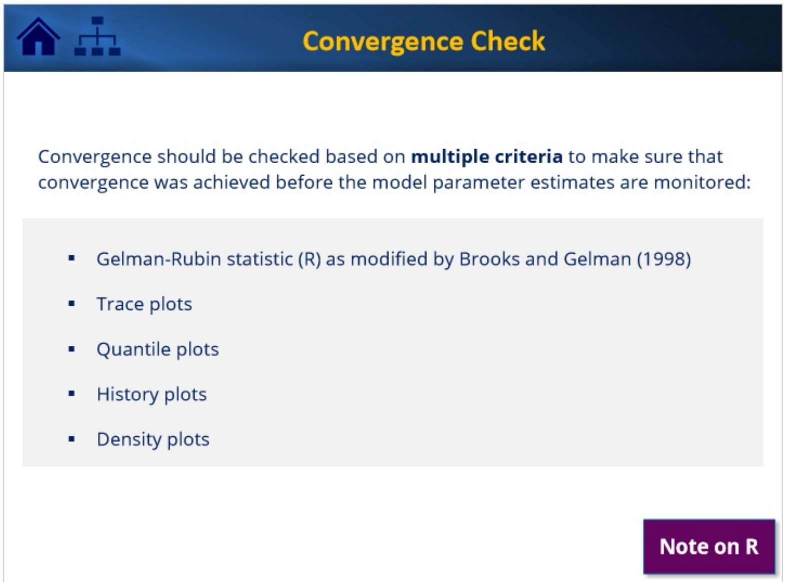
- **dnorm(** μ **,** τ **)** is the normal distribution or truncated normal distribution with parameters μ and $\tau = 1/\sigma^2$, (0, 0.001), as flat prior: θ , b
- **dnorm(** μ **,** τ **) I(L, U)** is the truncated normal distribution with parameters μ and $\tau = 1/\sigma^2$: a
- **Lognorm (** μ **,** τ **)** is the log-normal distribution with parameters μ and $\tau = 1/\sigma^2$, (0, 0.001), as flat prior: a
- **dbeta(**a**,**b**)** is the beta distribution with parameters a and b , (0.001, 0.001), as flat prior: c , d
- **dgamma(**a**,**s**)** is the gamma distribution, (0.001, 0.001), as flat prior: σ^2

4.21 Bookend: Prior Specification

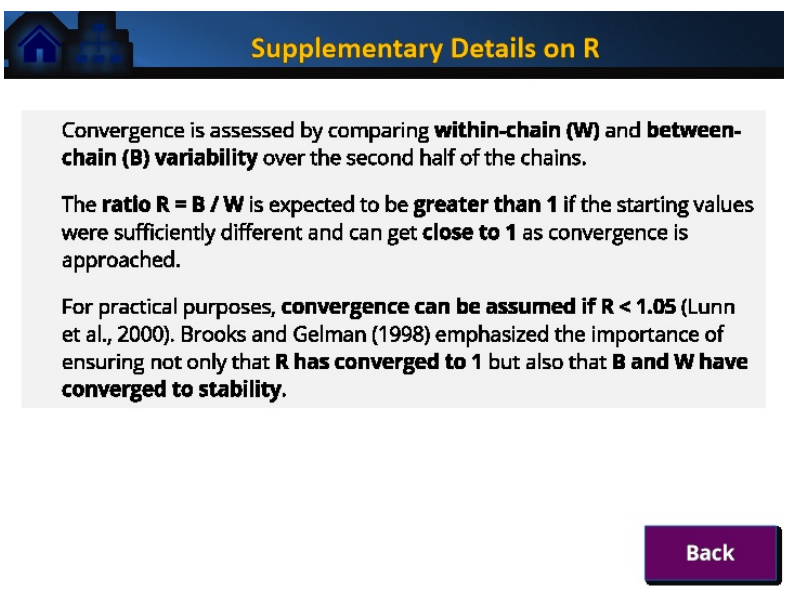


4.22 Bookmark: Convergence Checks

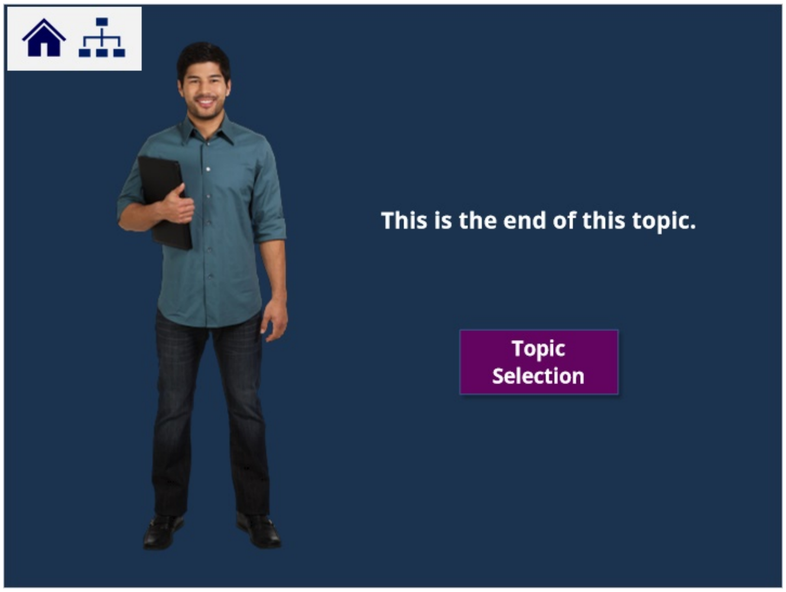
4.23 Convergence Checks



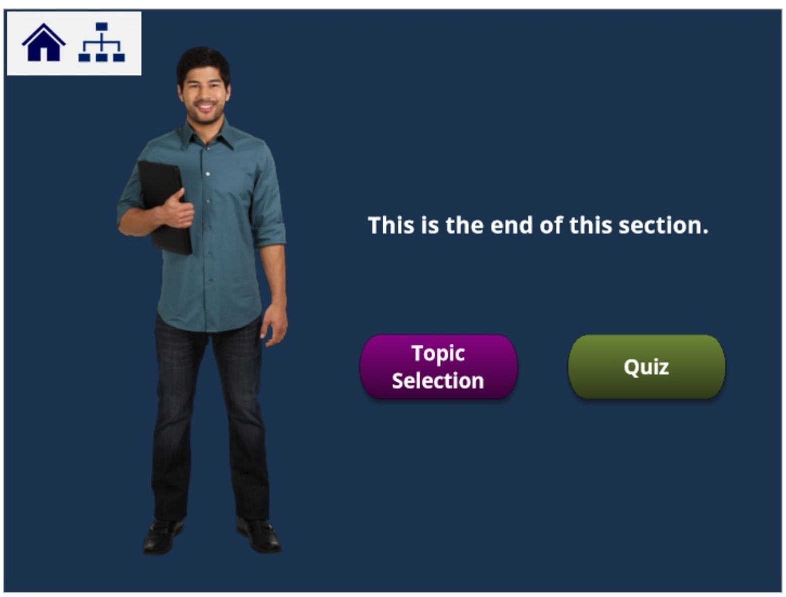
Details on R (Slide Layer)



4.24 Bookend: Convergence Checks

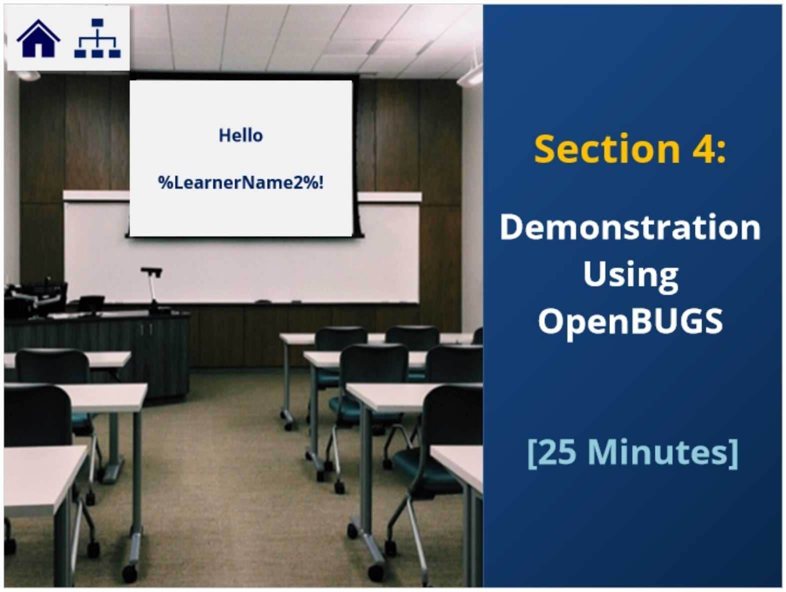


4.25 Bookend: Section 4

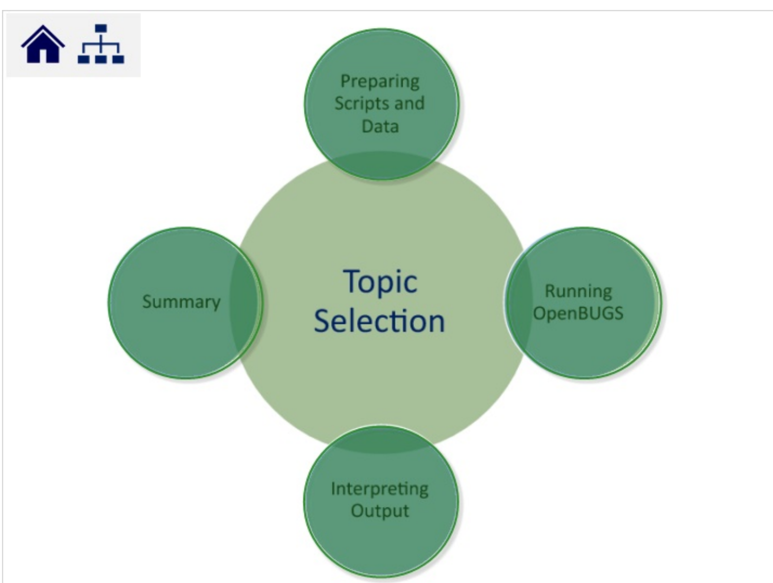


5. Section 4: Demonstration Using OpenBUGS

5.1 Cover: Section 5

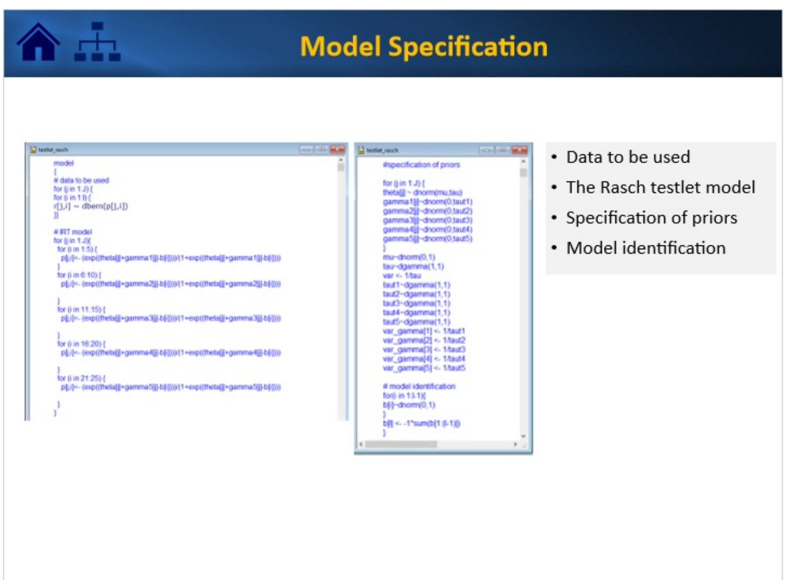


5.2 Topic Selection

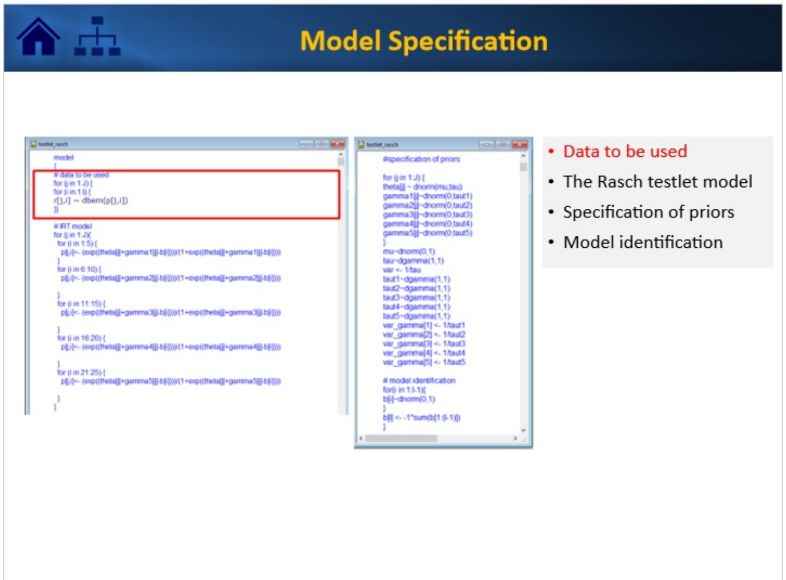


5.3 Bookmark: Preparing Script and Data

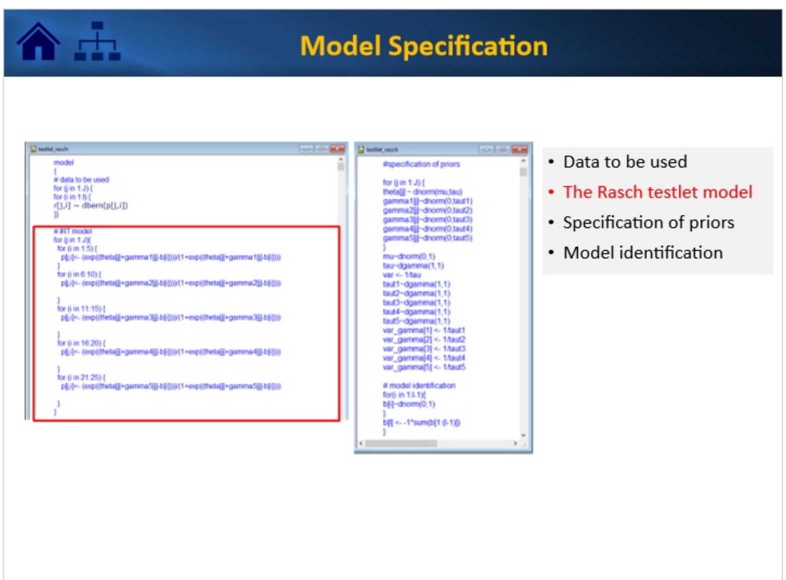
5.4 Model Specification I



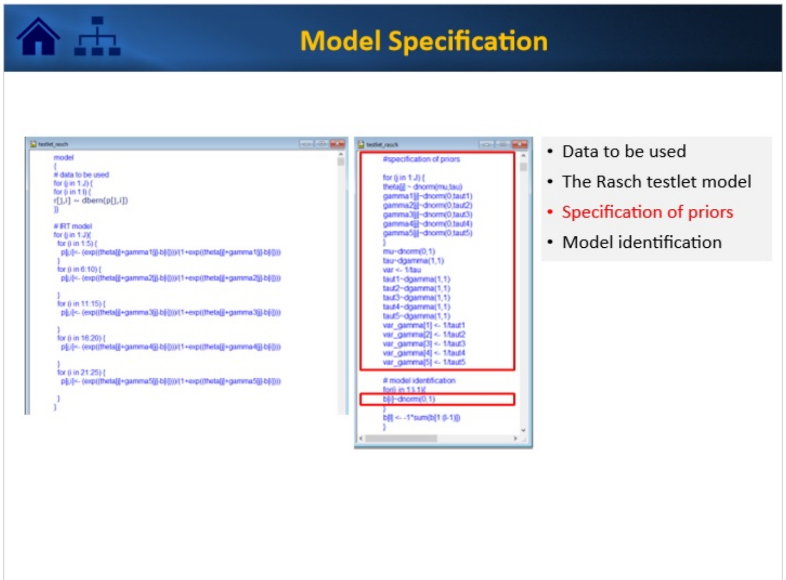
5.5 Model Specification III



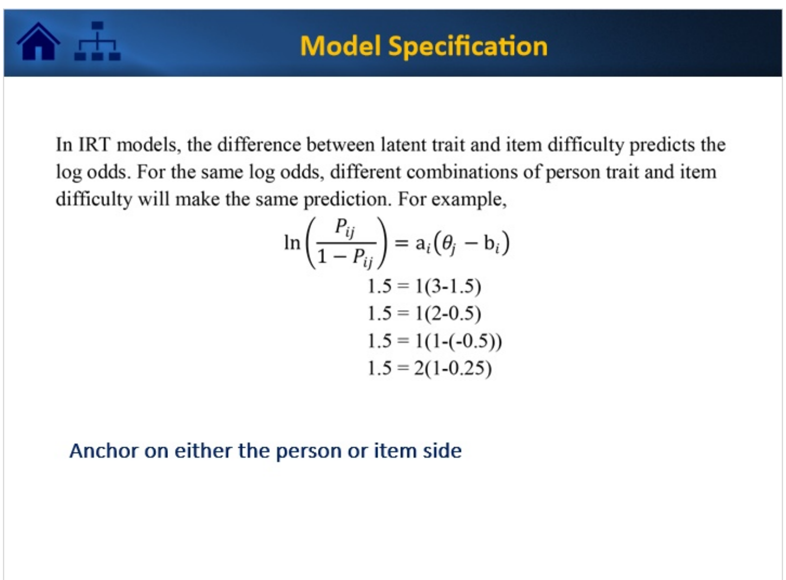
5.6 Model Specification III



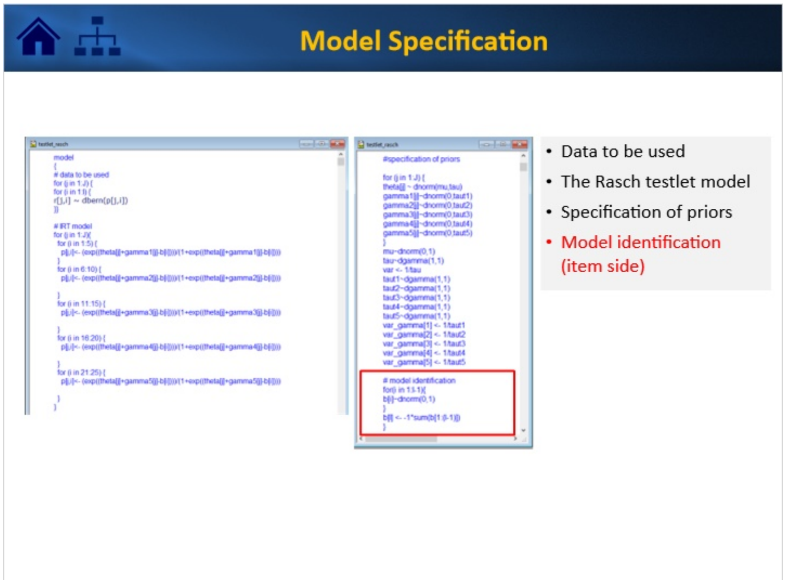
5.7 Model Specification IV



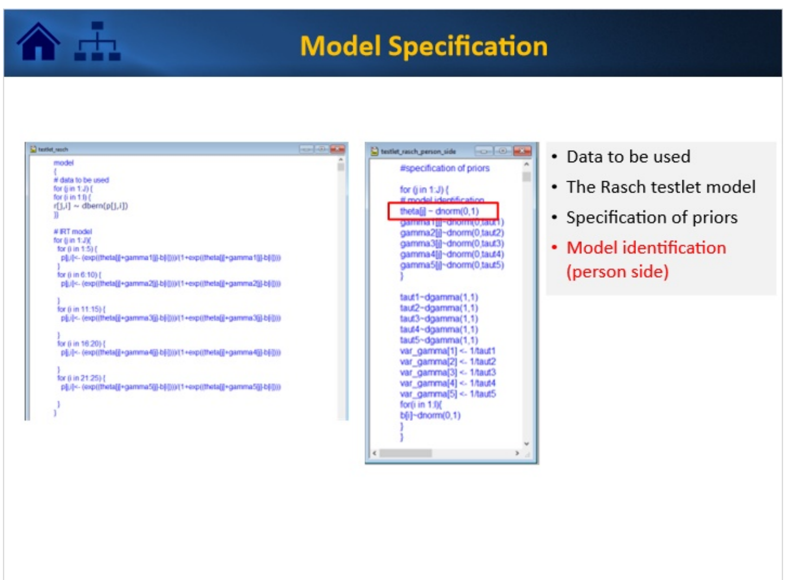
5.8 Model Specification V



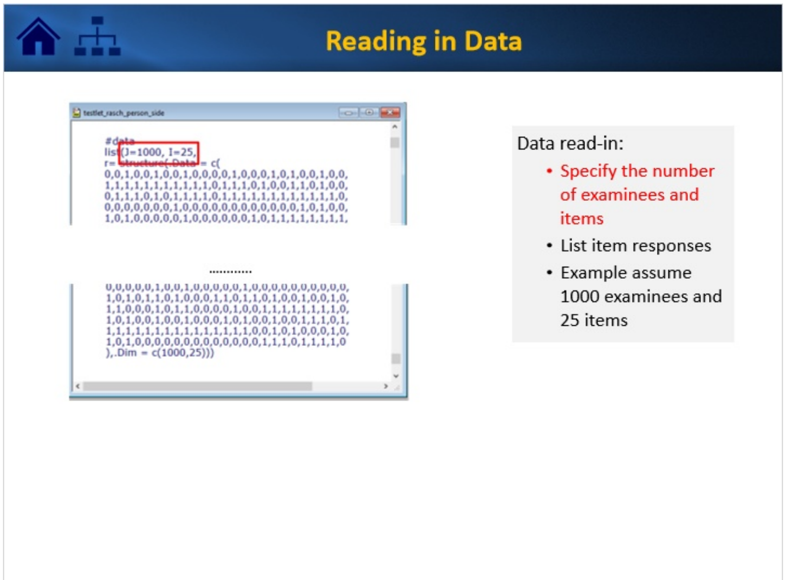
5.9 Model Specification VI



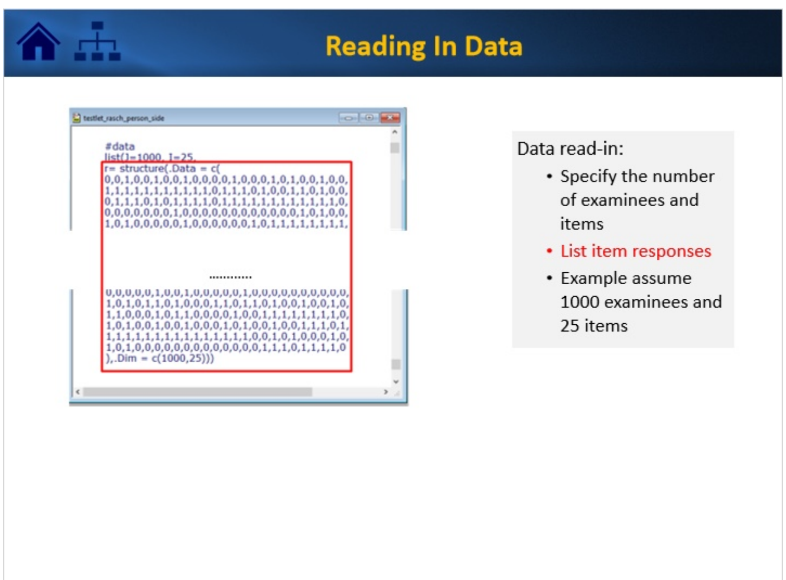
5.10 Model Specification VII



5.11 Reading in Data I



5.12 The Partial Credit Testlet Model I



5.13 The Partial Credit Testlet Model I

The Partial Credit Testlet Model

Assume four score categories: 0, 1, 2, 3
Recode scores into 1, 2, 3, 4

The probability of getting a certain score is:

$$t_1 = \exp(\theta_j + \gamma_{jd(i)} - b_i - d_{i1}), \quad t_2 = \exp(2\theta_j + 2\gamma_{jd(i)} - 2b_i - d_{i1} - d_{i2}),$$

and

$$t_3 = \exp(3\theta_j + 3\gamma_{jd(i)} - 3b_i - d_{i1} - d_{i2} - d_{i3})$$

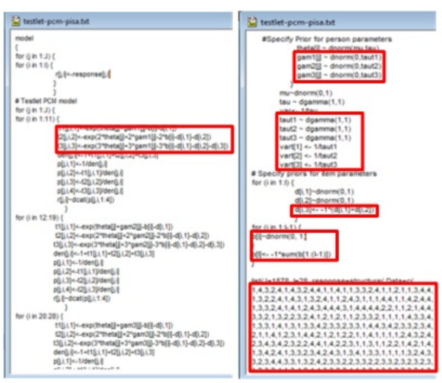
then:

$$pr(X_{ji} = 0) = \frac{1}{1 + t_1 + t_2 + t_3} \qquad pr(X_{ji} = 1) = \frac{t_1}{1 + t_1 + t_2 + t_3}$$

$$pr(X_{ji} = 2) = \frac{t_2}{1 + t_1 + t_2 + t_3} \qquad pr(X_{ji} = 3) = \frac{t_3}{1 + t_1 + t_2 + t_3}$$

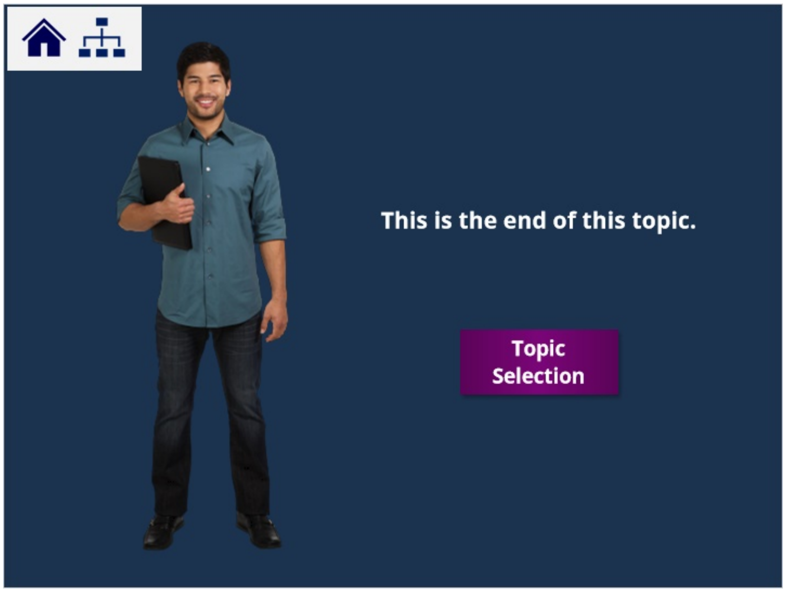
5.14 The Partial Credit Testlet Model II

The Partial Credit Testlet Model



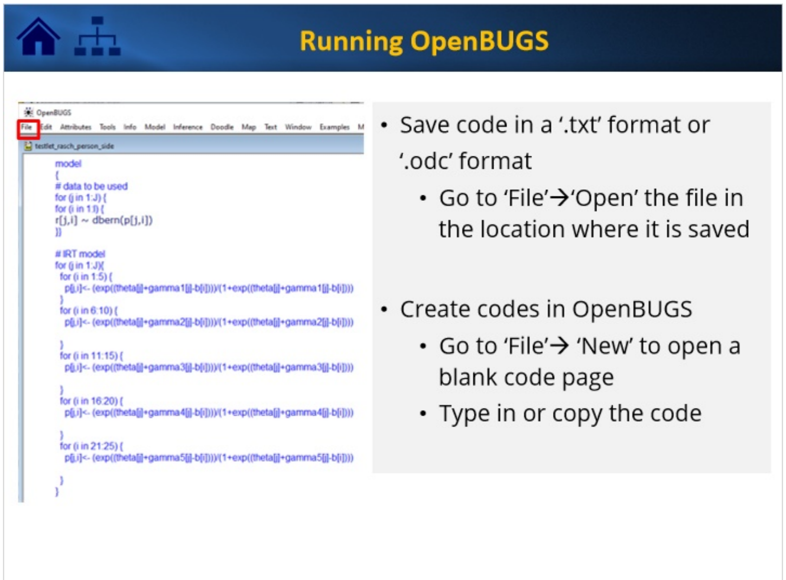
- Testlet variances
  - Items nested with testlets as
    - Items 1 to 11----testlet 1
    - Items 12 to 19----testlet 2
    - Items 20 to 28----testlet 3
  - Testlet effect parameters follow a normal distribution with a mean of 0 an a standard deviation to be estimated
  - Testlet variances follow an inverse gamma distribution.
- Model identification
  - Item side
  - Step parameters to be constrained
- Item responses
  - Starts at 1

5.15 Bookend: Preparing the Script and Data

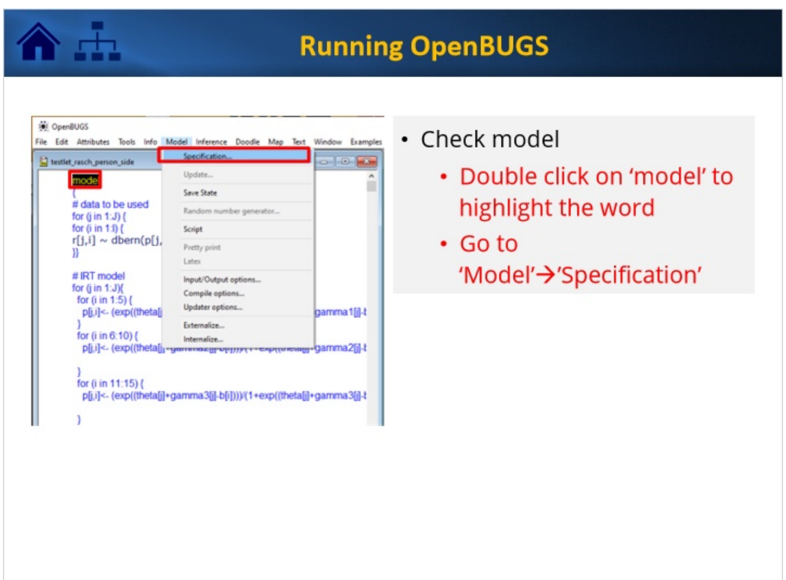


5.16 Bookmark: Running OpenBUGS

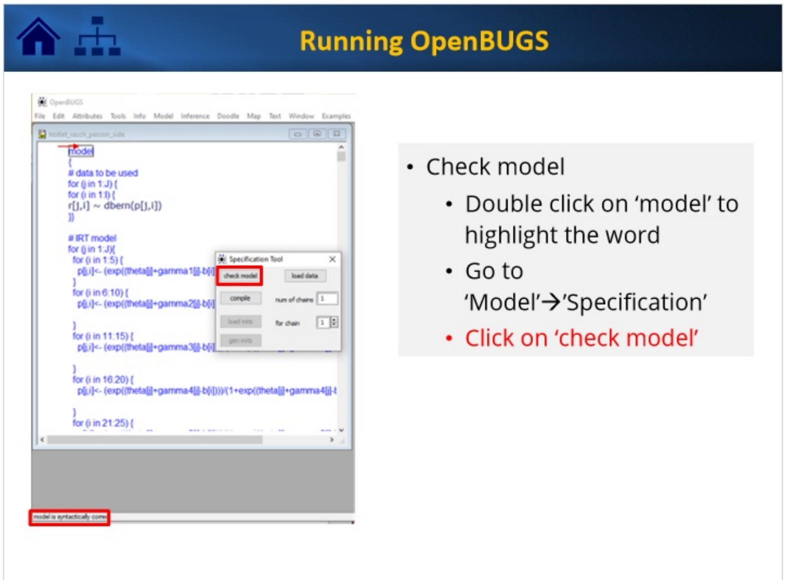
5.17 Running OpenBUGS I



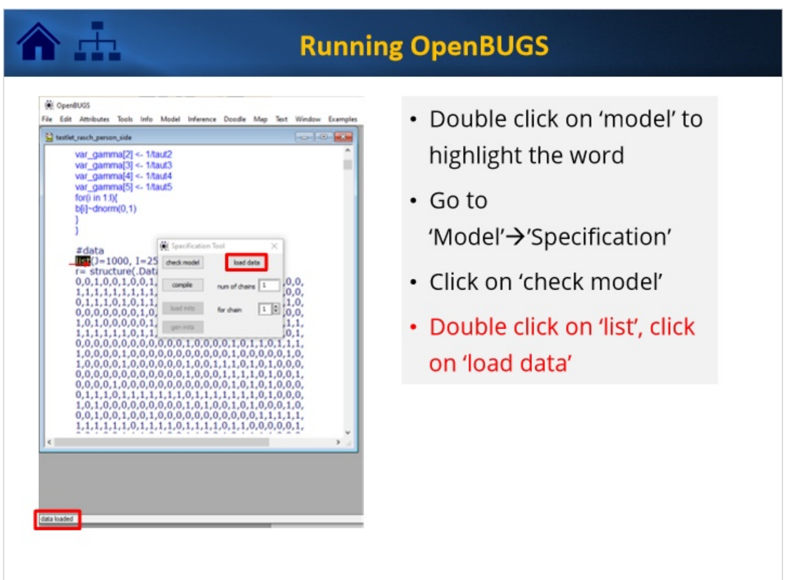
5.18 Running OpenBUGS II



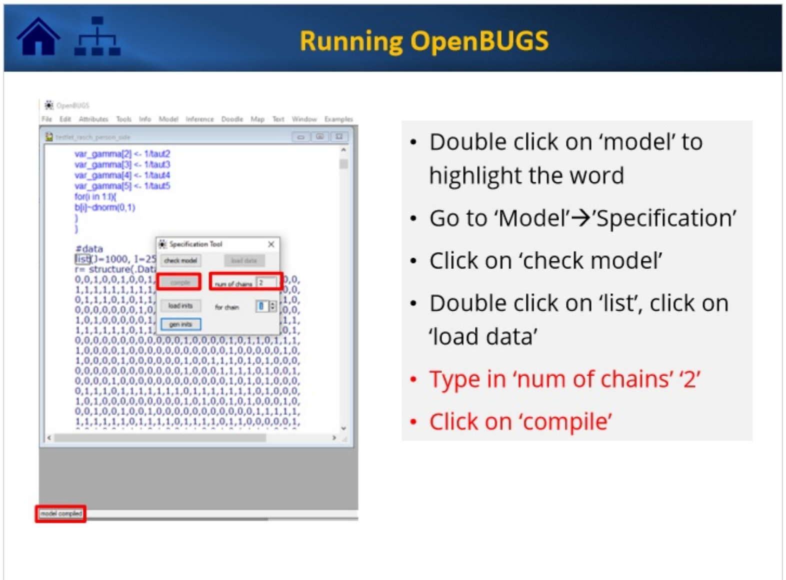
5.19 Running OpenBUGS III



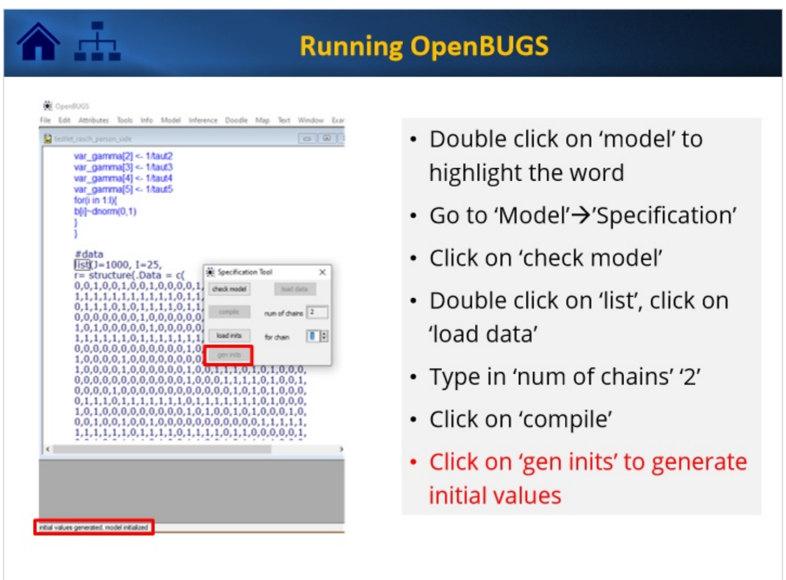
5.20 Running OpenBUGS V



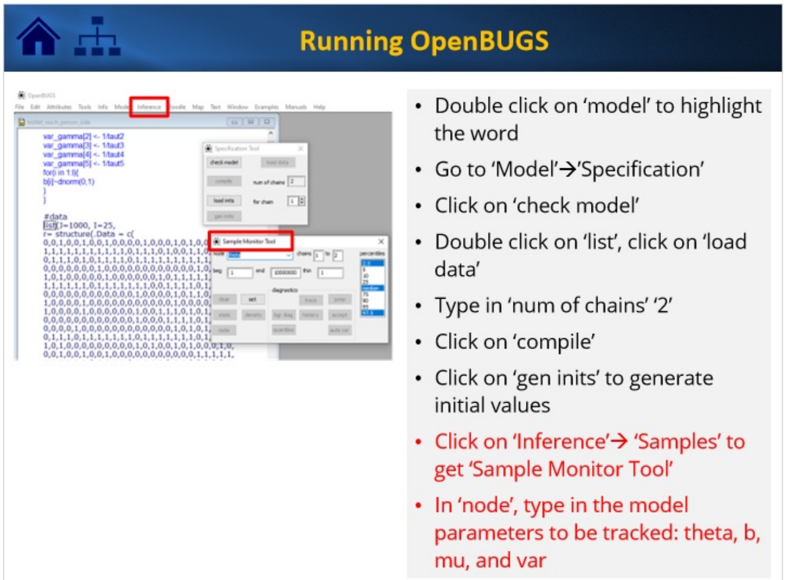
5.21 Running OpenBUGS VI



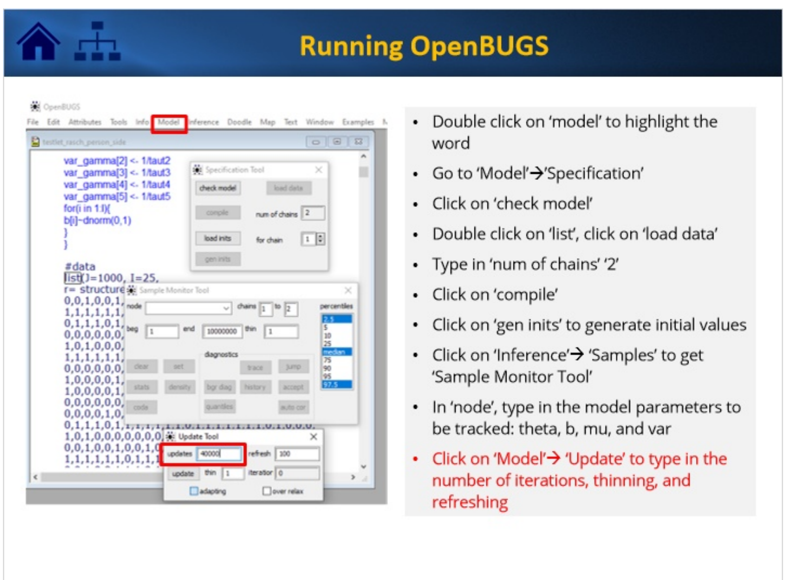
5.22 Running OpenBUGS VII



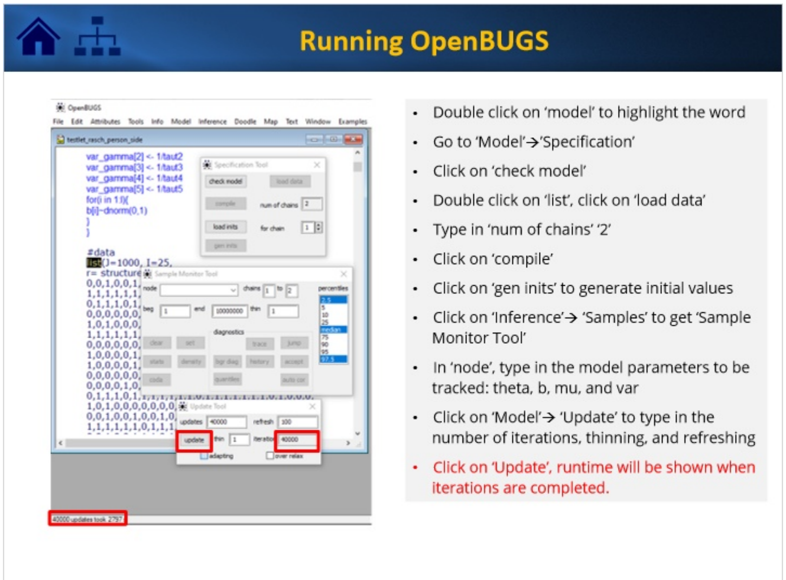
5.23 Running OpenBUGS VIII



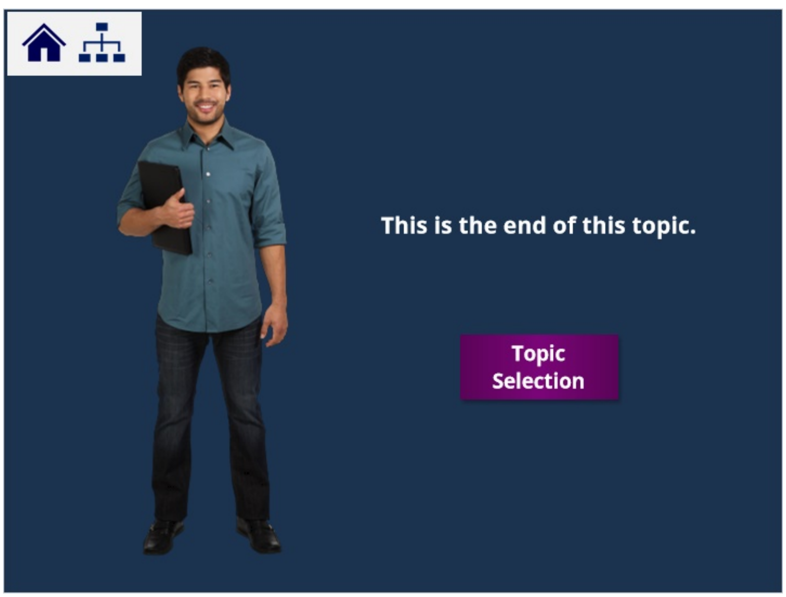
5.24 Running OpenBUGS IX



5.25 Running OpenBUGS X

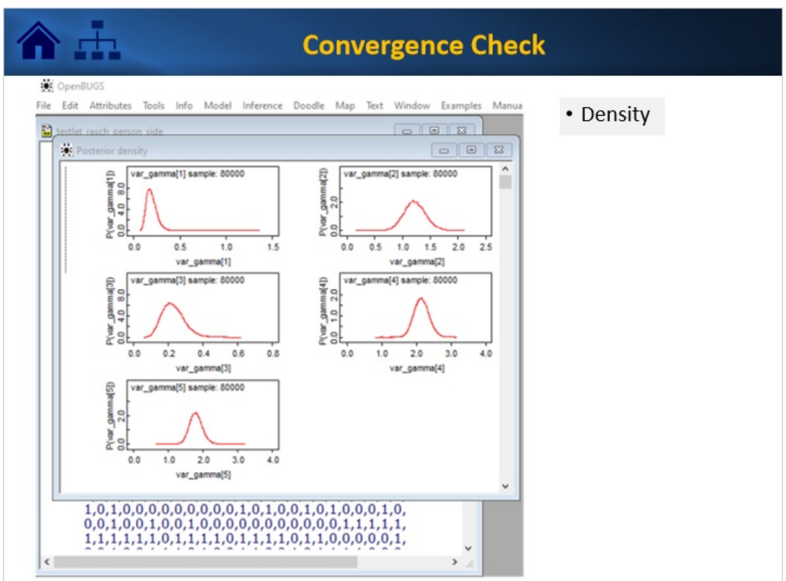


5.26 Bookend: Running OpenBUGS

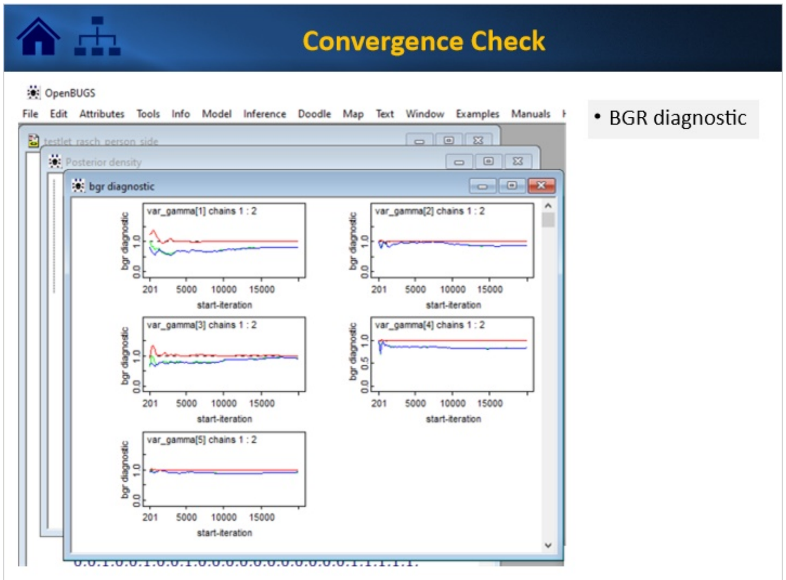


5.27 Bookmark: Interpreting Output

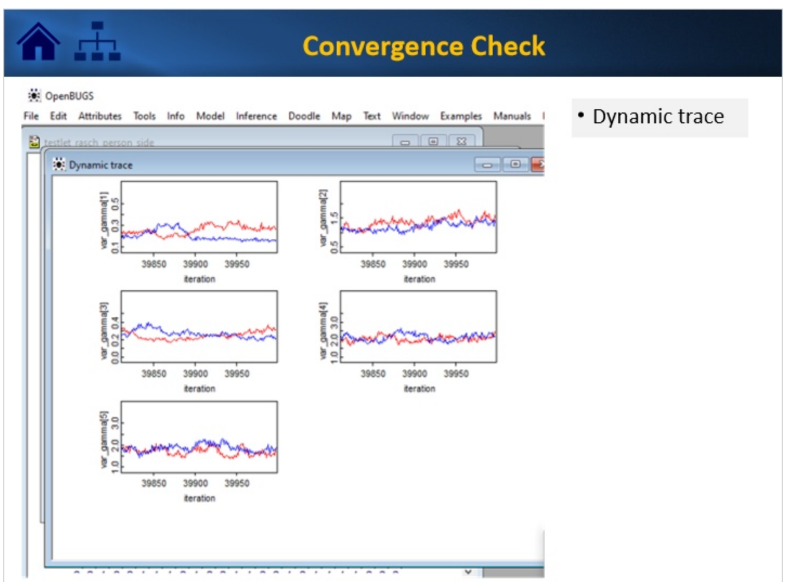
5.28 Convergence Check I



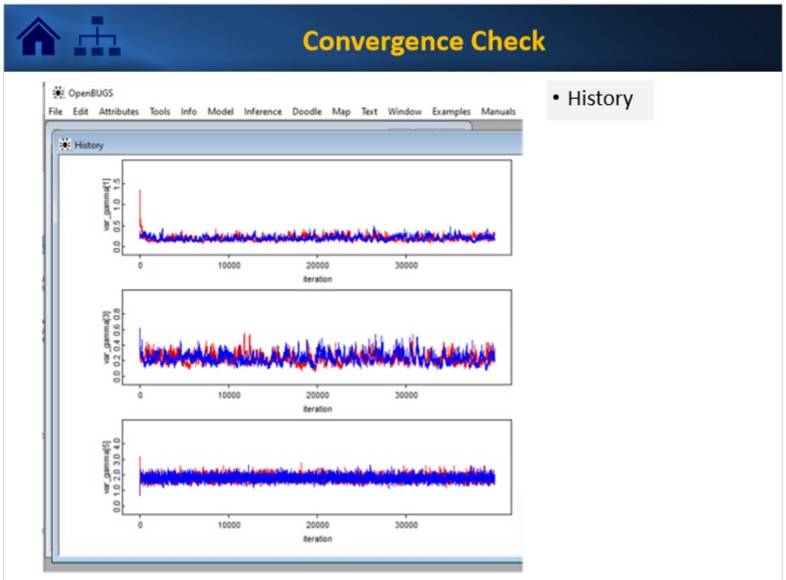
5.29 Convergence Check II



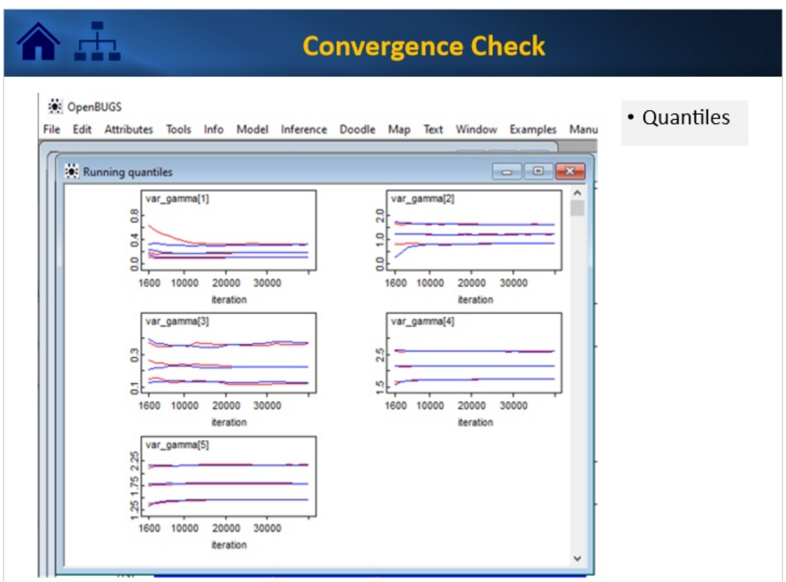
5.30 Convergence Check III



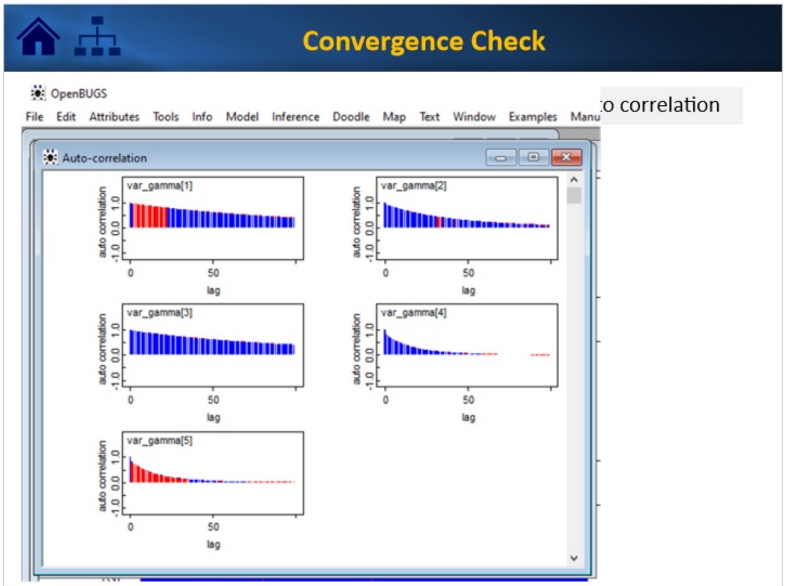
5.31 Convergence Check IV



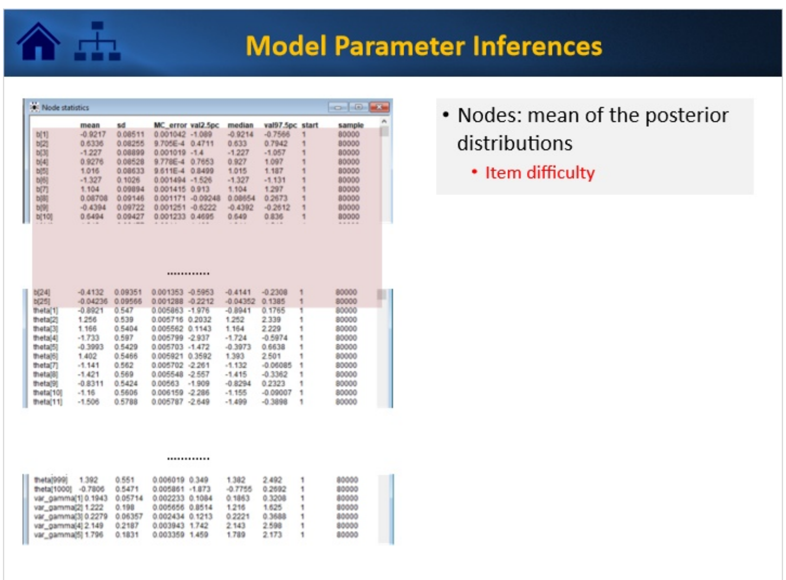
5.32 Convergence Check V



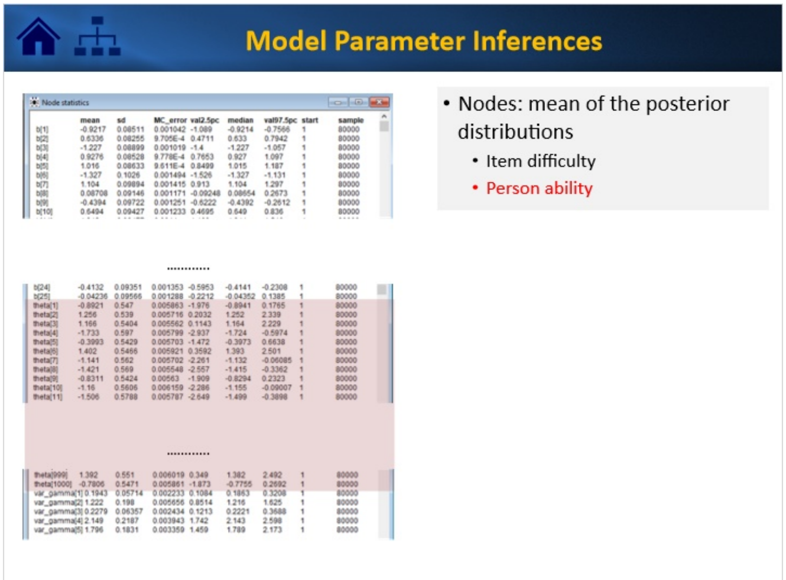
5.33 Convergence Check VI



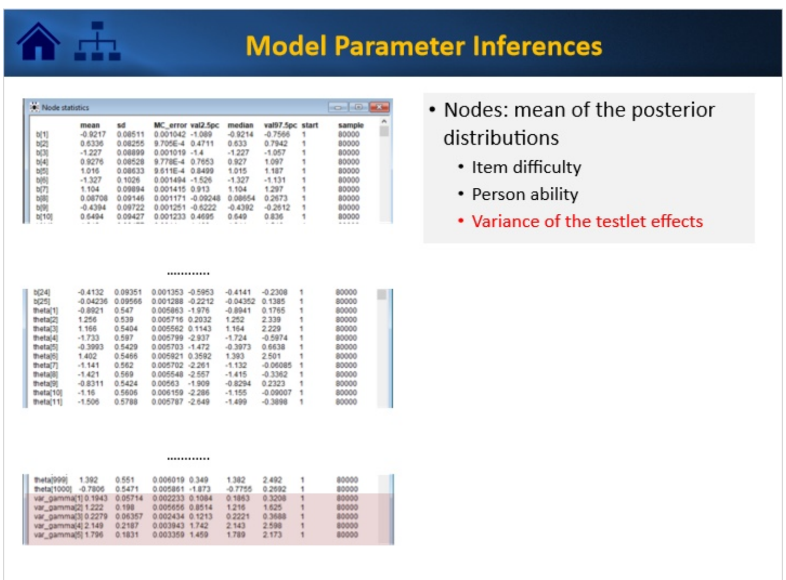
5.34 Model Parameter Inferences I



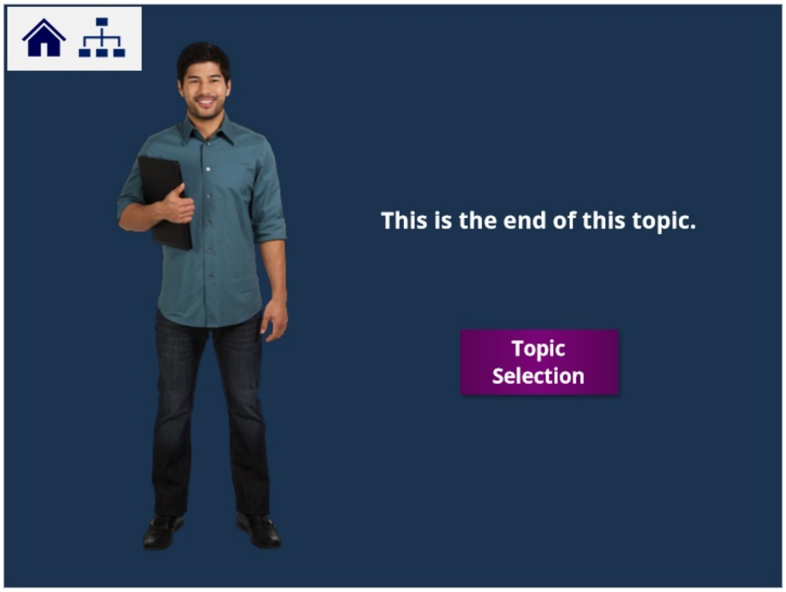
5.35 Model Parameter Inferences II



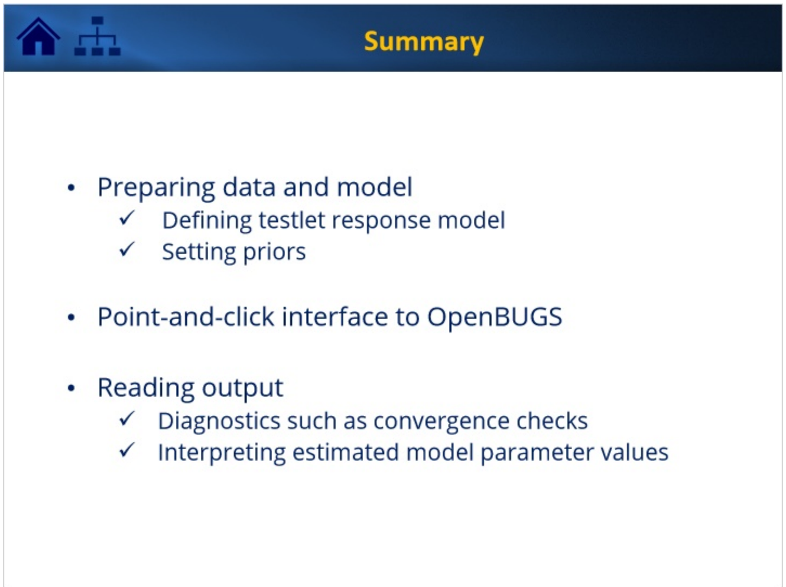
5.36 Model Parameter Inferences III



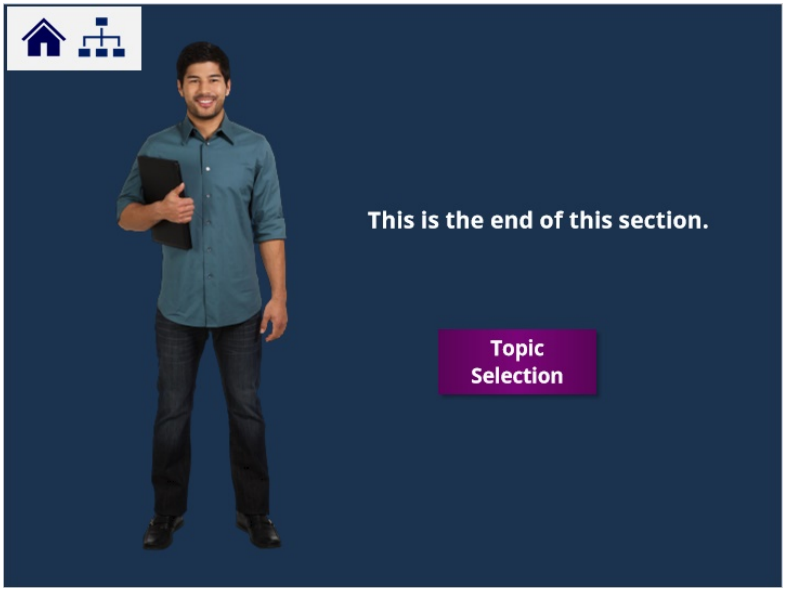
5.37 Bookend: Interpreting Output



5.38 Summary



5.39 Bookend: Section 5



5.40 Module Cover (END)